JAMES JOSEPH SYLVESTER.

Address delivered at a memorial meeting at the Johns Hopkins University, May 2, 1897,

BY DR. FABIAN FRANKLIN.

We have come together to do honor to the memory of the great man whose work in initiating and for seven years conducting the mathematical department of this institution, will always remain one of the proudest traditions of the John Hopkins University. To me, as one who was long his pupil, and who owes so much to his inspiration, has been assigned the task of saying something about the work and the genius of Sylvester, and especially about the influence which he exerted, while in Baltimore, upon the study of mathematics here and upon the advancement of mathematical research in America.

Since his death, there has appeared in the English journal *Nature*, and has been reprinted in the Johns Hopkins Circulars, a review of his life and work by Major McMahon; and in 1889, when that work was well-nigh ended, Sylvester's great compeer and friend, Professor Cayley, contributed to the columns of the same journal a sketch of Sylvester's labors. One of his Baltimore pupils, too, Professor Halsted of the University of Texas, has given in *Science* an account of his life and achievements. It is therefore the less necessary to undertake here to give anything in the nature of an enumeration of even his most signal contributions to mathematics.

His influence upon the development of mathematical science rests chiefly, of course, upon his work in the theory of invariants. Apart from Sir William Rowan Hamilton's invention and development of quaternions, this theory is the one great contribution made by British thought to the progress of pure mathematics in the present century, or indeed since the days of the contemporaries of Newton. From about the middle of the eighteenth century until near the middle of the nineteenth, English mathematics was in a condition of something like torpor. The second half of the eighteenth century was one of the most brilliant periods in the history of mathematics; but the magnificent achievements of Euler, Lagrange, Laplace awakened no response on the other side of the narrow seas. It seems almost incredible that the complacent conservatism of Cambridge went so far that even the notation of mathematical analysis

as used on the Continent was untaught there until about 1820. Babbage tells us, in his "Passages from the Life of a Philosopher," how he, together with Herschel, Peacock and a few others, founded in 1812 the "Analytical Society" for promoting (as Babbage humorously expressed it) "the Principles of D-ism in opposition to the Dot-age of the University." It is from the translation by these three men (in 1816), of Lacroix's Treatise on the Differential and Integral Calculus, together with the publication by them, four years later, of two volumes of illustrative examples, that the first impulse toward a revival of mathematics in England is usually dated. Nothing could show more thoroughly the insular and retrograde condition of English mathematics in the early part of this century. The sticking to Newton's fluxions and dots, and the barring out of Leibnitz's differentials and d's, may be set down as a consequence of the great Newton-Leibnitz controversy; but, whatever the cause, so complete a separation from the great current of European thought implies stagnation deep-seated and not easily to be removed. And accordingly, it proved to be the case that in the magnificent extension of the bounds of mathematics which was effected by the continental mathematicians during the first four decades of the present century, England had no share. It is almost literally correct to say that the history of mathematics for about a hundred years might be written without serious defect with English mathematics left entirely out of account.

That the like statement cannot be made in regard to the past fifty years is due preëminently to the genius and labors of three men: Hamilton, Cayley and Sylvester. Hamilton was a high and solitary genius, who constructed and developed, unaided, a great mathematical method. Great as was this work, it lay so entirely apart from the general line of research that it did not, in his own time at least, awaken widespread activity on the part of others either at home or abroad. On the other hand, the theory of invariants had a history of what may be called the normal type. Its origin is to be found in Boole's discoveries of isolated instances of invariance; these led Cayley to institute a systematic investigation of this remarkable and significant phenomenon, and Cayley's researches awakened the ardent interest of Sylvester. Under the hands of these two great masters, a new and important province was rapidly added to the domain of algebra. Not only did the English mathematicians join in the work, but Hermite in France, Aronhold and Clebsch in Germany, Brioschi in Italy, and other continen-

tal mathematicians seized upon the new ideas, and the theory of invariants was for three decades one of the leading objects of mathematical research throughout Europe. It is impossible to apportion between Cayley and Sylvester the honor of the series of brilliant discoveries which marked the early years of the theory of invariants. Their names are linked together as the creators of a new and beautiful development of algebra, the ideas of which have profoundly influenced the progress also of geometry and of analysis generally. "The Theory of Invariants," says MacMahon, "sprang into existence under the strong hand of Cayley, but that it emerged finally a complete work of art, for the admiration of future generations of mathematicians, was largely owing to the flashes of inspiration with which Sylvester's intellect illuminated it." It is pleasant to know that the triumphs of neither were marred by any dispute as to personal claims or by anything even approaching jealousy. On the contrary, these two men of genius, antipodes of each other in temperament and habits of work, were alike in the constancy of their mutual friendship, regard and admiration.

I have dwelt thus long on Sylvester's connection with the creation of the theory of invariants because it is by that chiefly that he left his trace upon the history of mathematics in its large outlines. But his genius is quite as strikingly shown in researches of a more isolated character. Ten years before the date of his work in invariants, he wrote in quick succession several remarkable memoirs on algebraic subjects, especially on Sturm's functions and on elimination. His researches in the theory of partitions of numbers are among the most original and remarkable of his works. He made important and striking additions to the theory of matrices. In the theory of numbers, he was especially interested in ternary cubic forms. The question of the distribution of prime numbers had a great fascination for him; and he succeeded, while in Baltimore, in making an impression upon this recondite problem in that he contracted the limits found by Tchebycheff for the number of primes contained within a given range. His work seldom touched on geometry, but his "Theory of Reciduation" in connection with cubic curves is a beautiful structure, to which he made some remarkable additions while in Baltimore. I am not, however, attempting to give a survey of his work; suffice it to add that, in addition to the subjects named, he made contributions to astronomy, to dynamics and to the theory of link-motion, besides other special subjects.

One of the most striking of Sylvester's achievements was his demonstration and extension of Newton's unproved rule concerning the number of imaginary roots of an algebraic equation. Newton had left no trace of the process of thought by which he had arrived at his rule, nor had he given any indication of the basis on which it rests. All attempts of later mathematicians to establish it had proved futile. It was characteristic of Sylvester to set himself the task of filling up this lacuna in mathematics. The things that attracted and fascinated him were of two kinds, which may be called opposite to each other. On the one hand, he revelled in any new and prolific method; the feeling of creation, of abounding productiveness, was to him as the breath of his nostrils. It was largely this that made the Theory of Invariants so congenial to him. To see a whole new world, full of unexpected and harmonious relations, expanding before him, was to fill him with an absorbing and exuberant enthusiasm. In the case of invariants, it may be said that his joy in this sense of creation was not even confined to the discovery of theorems; the algebraic forms themselves were to him as living beings, and the processes, invented largely by himself, for causing these creatures of the mathematical intellect to generate their kind, were to him a source of genuine delight.

Alongside of this love of prolific creation, another intellectual bent, on the surface at least of quite the opposite character, was equally strongly marked in Sylvester. Any crucial problem, especially one that was associated with the name of one of the great masters, if once it attracted Sylvester's attention, fastened itself upon his mind with a grip that seemed never to slacken its tenacity. It kept coming up, again and again, for years, and, as long as it remained unsolved, seemed to become periodically a source of unrest and discomfort to his mind. He had not the serenity which belonged to many other great mathematicians, and notably to Cayley, and which, in a great measure, permitted them to choose among the possible subjects of thought such as they deemed most profitable to pursue. With Sylvester, such tranquil and deliberate choice was entirely out of the question. His temperament was essentially poetic, and it would have been as impossible for him to concentrate the powers of his mind on one subject when the current of his thought was setting toward another as it would have been for Burns to decide in cold blood to write a poem like "Highland Mary" or "The Daisy" when the inspiration of "Tam O'Shanter" was upon him.

It was the mention of Sylvester's demonstration of Newton's rule that suggested these reflections. We, who knew him well in later years, can find no difficulty in understanding the hold this problem had upon him. It was the good fortune of his early hearers in this university to be present when he came into the lecture-room, flushed with the achievement of a somewhat similar task. A certain fundamental theorem in the theory of invariants, which had formed the basis of an important section of Cayley's work, had never been completely demonstrated. The lack of this demonstration had always been, to Sylvester's mind, a most serious blemish in the structure. He had, however, he told us, years ago given up the attempt to find the proof as hopeless. But, upon coming fresh to the subject in connection with his Baltimore lectures, he again grappled with the problem, and, by a fortunate inspiration, succeeded in solving it. It was with a thrill of sympathetic pleasure that his young hearers thus found themselves in some measure associated with an intellectual feat by which had been overcome a difficulty that had successfully resisted assault for a quarter of a century. Nor was this the only instance in which we had an opportunity of observing the tenacious hold upon his intellect of any problem that had come to assume in his mind the aspect of a challenge to the powers of mathematicians.

I have said that Sylvester's powers were set in motion by two opposite kinds of stimulus; that of abundantly rewarding results, and that of the stubborn resistance of concentrated difficulty. In both these kinds of endeavor, he achieved many and signal triumphs. That intermediate kind of effort which slowly and patiently builds up and improves and perfects one's own work, and which gives minute and prolonged study to the work of others, he did not command in any notable degree. He seemed incapable of reading mathematics in a purely receptive way. Apparently a subject either fired in his brain a train of active and restless thought, or it could not retain his attention at all. To a man of such a temperament, it would have been peculiarly helpful to live in an atmosphere in which his human associations would have supplied the stimulus which he could not find in mere reading. The great modern work in the theory of functions and in allied disciplines, he never became acquainted with. No one who witnessed the flaming up of his energies when, at the age of 62, in Baltimore, he felt himself, for the first time, among a band of enthusiastic young workers pursuing pure mathematics for its own

sake, can doubt what the effect would have been if, in the prime of his powers, he had been surrounded by the influences which prevail in Berlin or in Göttingen. It may be confidently taken for granted that he would have done splendid work in those domains of analysis which have furnished the laurels of the great mathematicians of Germany and France in the second half of the present century.

Cambridge, his natural intellectual home, would have been far less helpful, since it was examinations and not research which gave tone to the mathematical life there. But Cambridge would of course have been immeasurably better than the situations in which he actually found himself for forty years after his winning of the Second Wranglership. From a career at Cambridge, to the great loss of that University, of himself, and of mathematics, he was debarred by the religious tests then obtaining in the old English universities. Professor Halsted, in his account of Sylvester's work already referred to, points out how the vicissitudes of his career were reflected in the richness or the meagreness of his mathematical production from period to period.

The life and work of Sylvester illustrate in a striking way the futility of the dispute as to the relative importance of native qualities and of external circumstances in determining the achievements of great men. If any man was ever an original genius, with consuming ardor for one intellectual pursuit, with love and devotion to it burning in youth and undiminished in age, Sylvester was such a man. If any province of thought is open to every worker in it, to work in as he pleases, uninfluenced by the doings of those who happen to be in his neighborhood, in his university, in his country, one would say that mathematics is that province. Yet no one could know Sylvester without feeling that, great and original as was his genius, environment must, in his case, exercise an extraordinary influence on its activity. He was sensitive, passionate, fiery; the glowing language in which he habitually indulged in the midst of his mathematical memoirs was but a reflection of his ardent and excitable temper. Such a man must needs be keenly subject to depression and exaltation, to fits of apathy and ardor, according to the nature of his surroundings and experiences. Those who knew him cannot fail to be convinced that, eminent as were his actual achievements, they do not afford a true measure of his mathematical powers, in comparison with those of his great contemporaries. For he was at once less advantageously circumstanced than they, and in an exceptional degree subject to the influence of his surroundings.

Of his work as a teacher, I can speak only upon the basis of his activity in this university. The one thing which constantly marked his lectures was enthusiastic love of the thing he was doing. He had in the fullest possible degree, to use the French phrase, the defect of this quality; for as he almost always spoke with enthusiastic ardor, so it was almost never possible for him to speak on matters incapable of evoking this ardor. In other words, the substance of his lectures had to consist largely of his own work, and, as a rule, of work hot from the forge. The consequence was that a continuous and systematic presentation of any extensive body of doctrine already completed was not to be expected from him. Any unsolved difficulty, any suggested extension, such as would have been passed by with a mention by other lecturers, became inevitably with him the occasion of a digression which was sure to consume many weeks, if indeed it did not take him away from the original object permanently. Nearly all of the important memoirs which he published while in Baltimore arose in this way. We, who attended his lectures, may be said to have seen these memoirs in the making. He would give us on the Friday the outcome of his grapplings with the enemy since the Tuesday lecture. Rarely can it have fallen to the lot of any class to follow so completely the workings of the mind of the master.

Not only were we thus privileged to see "the very pulse of the machine," to learn the spring and motive of the successive steps that led to his results, but we were set aglow by the delight and admiration which, with perfect naïveté, and with that luxuriance of language peculiar to him, Sylvester lavished upon these results. That in this enthusiastic admiration he sometimes lacked the sense of proportion cannot be denied. A result announced at one lecture and hailed with loud acclaim as a marvel of beauty was by no means sure of not being found before the next lecture to have been erroneous; but the Esther that supplanted this Vashti was quite certain to be found still more supremely beautiful. The fundamental thing, however, was not this occasional extravagance, but the deep and abiding feeling for truth and beauty which underlay it. No young man of generous mind could stand before that superb gray head and hear those expositions of high and dear-bought truths, testifying to a passionate devotion undimmed by years or by arduous labor, without carrying away that which ever after must give to the pursuit of truth a new and deeper significance in his mind.

As is well known, Sylvester had an extraordinary faculty

for the coinage of words, which, indeed, was merely a part of his remarkably keen sense for language in general. In this matter of the coining of words, he doubtless went to extremes, as he did in other things; but there can be no question of the great service he rendered to the new science of invariants by the creation of a whole vocabulary which rendered possible the crystallization of thought in what would otherwise have been a comparatively amorphous mass. There are doubtless other departments of mathematics which would be made more manageable by the skillful application of just such a name-creating faculty. Any mathematical conception with which Sylvester had much to do had to be equipped with a name. He justly felt that the absence of it impeded thought, and he could not be comfortable in this state of things. His hearers will not forget how, after getting along for some time with the notation $\varphi(n)$, by which mathematicians had been content, from the time of Legendre, to designate the number of numbers less than a given number and prime to it, he came into the lecture-room one afternoon, and began in his most emphatic manner, thus: "Gentlemen, I am about to introduce to you a name that has been struggling for birth for a century!" I may mention here an instance of his delicate sense for words—and indeed for things—which occurred during a walk I was taking with him. We were speaking of Mitchell, then a fellow in mathematics here, and I said that he impressed me as bearing a resemblance to Abraham Lincoln. He seemed struck with the idea, and, after a moment's silence, said "Yes, there is a certain not inelegant stiffness about him which reminds one of Lincoln." Where Sylvester got his impression of Lincoln I do not know; but surely it would have been difficult to hit off the outward effect of the man in words more accurately chosen.

Another direction which his talent for expression and his love of the niceties of language took was that of versification. He made some excellent translations from Horace and from German poets, besides writing a number of pieces of original verse. The tours de force in the way of rhyming, which he performed while in Baltimore, were designed to illustrate the theories of versification of which he gives indications in his little book called "The Laws of Verse." The reading of the Rosalind poem at the Peabody Institute was the occasion of an amusing exhibition of absence of mind. The poem consisted of no less than 400 lines, all rhyming with the name Rosalind (the long and short sound of the i both being allowed). The audience quite filled the

hall, and expected to find much interest or amusement in listening to this unique experiment in verse. But Professor Sylvester had found it necessary to write a large number of explanatory footnotes, and he announced that in order not to interrupt the poem he would read the footnotes in a body first. Nearly every footnote suggested some additional extempore remark, and the reader was so interested in each one that he was not in the least aware of the flight of time, or of the amusement of the audience. When he had dispatched the last of the notes, he looked up at the clock, and was horrified to find that he had kept the audience an hour and a-half before beginning to read the poem they had come to hear. The astonishment on his face was answered by a burst of good-humored laughter from the audience; and then, after begging all his hearers to feel at perfect liberty to leave if they had engagements, he read the Rosalind poem.

Sylvester was quick-tempered and impatient, but generous, charitable and tender-hearted. He was always extremely appreciative of the work of others and gave the warmest recognition to any talent or ability displayed by his pupils. He was capable of flying into a passion on slight provocation, but he did not harbor resentment, and was always glad to forget the cause of quarrel at the earliest opportunity. I have it on extremely good authority that, in his intercourse with Professor Cayley, toward whom he maintained a life-long and devoted friendship, and his admiration of whom might be said to amount to reverence, little episodes of this kind were not absent. Some fancied injury would lead Sylvester to write Cayley an angry letter; Cayley, who was as serene and tranquil as Sylvester was passionate and excitable, would quietly leave the letter unanswered. In a few days another letter was sure to come from Sylvester, written as though nothing whatever had happened. The mention of Cayley leads me to recall an incident of the farewell reception given to Cayley in Hopkins Hall at the close of his residence here, which affords another illustration of Sylvester's felicity of expression. The platform was abundantly decorated with flowers, and Cayley, who was extremely shy and retiring, looked very uncomfortable in his conspicuous position upon it while Sylvester was speaking. Referring to Cayley's modesty, Sylvester suddenly turned toward him and said, "There he sits, like a victim decked with flowers!"

Sylvester did not, I believe, like to speak about religion. He was born a Jew, and was buried in the Jewish cemetery at Dalston. I am sure that he would not have subscribed

to any formulated creed, but he was a man of truly reverent mind and a sincere theist. It was notable that, in speaking of his own designs for the future, he quite habitually used the phrase "please God," with an accent that showed it was not a mere form of words. Once, when I asked him what was his estimate of Clifford, he said, with great earnestness: "Clifford is a very great genius; I only wish he would stick to mathematics instead of talking atheism."

Of Sylvester's influence upon this university, not only through his teaching, through the foundation of the *American Journal of Mathematics*, and through the constant stimulation of mathematical interest here by his incessant productiveness, but also through the infection of his enthusiasm, which was felt in every department of the university, it would be impossible to speak too strongly. His aggressive and singular personality seemed to act the part of a ferment which spread itself through the entire body of the university. In its prosperity and progress and fame he took the deepest interest, and his attachment to it was not weakened when he returned to his native land at the call of the University of Oxford.

Professor Sylvester's residence at the Johns Hopkins University constitutes an episode quite unique in the history of mathematics and education. Up to the time when he came to America, the study of the higher pure mathematics may be said, with almost literal truth, to have been non-existent in our country. He came, a man who had almost filled out what is usually spoken of as the alloted span of life, and at once inspired zeal and activity in a field which had been left almost uncultivated among us. The earliest outward effect of his ardor was the foundation of the *American Journal of Mathematics*, the first mathematical journal of any importance ever published in America, and almost the first journal devoted to any scientific specialty. It may truly be looked upon as the father of that army of scientific journals which have since overspread the country and testified to the growth of the higher learning among us. The prestige of his name and the fertility of his work could not do otherwise than excite emulation in other American centres of learning. While there doubtless would, in any case, have been progress in this direction, it must be set down as preëminently the result of Sylvester's presence in Baltimore that mathematical science in America has received the remarkable impetus which the last twenty years have shown. American names are no longer absent from the record of mathematical progress. We have not yet pro-

duced one of the heroes of mathematics; but there are now among us a dozen universities in each of which something, be it much or little, is being added to that splendid monument of human thought which bears the record of conquests made by so many of the intellectual giants of our race.

Among these giants Sylvester has without question the right to be reckoned. In the history of mathematics, his place will not be with the very greatest; but his work, brilliant and memorable as it was, affords no true measure of his intellectual greatness. Those who came within the sphere of his personality, could not but feel that, through the force of circumstances combined with the peculiarities of his poetic temperament, his performance, splendid as it was, has not adequately reflected his magnificent powers. Those of us who were connected with him cherish his memory as that of a sympathetic friend and generous critic. And in this university, as long as it shall exist, he will be remembered as the man whose genius illuminated its early years, and whose devotion and ardor furnished the most inspiring of all the elements which went to make those years so memorable and so fruitful.

HYPERBOLEA AND THE SOLUTION OF EQUATIONS.

BY MR. C. H. HINTON.

In the following pages, after a few remarks on the system of mathematics in vogue in Hyperbolea, I wish to show that a consideration of the methods of the Hyperboleans leads to a graphical representation of quantities by which, given an appropriate train of mechanism, not only the real, but also the imaginary roots of an equation can be mechanically found.

Hyperbolea is a land in which distance is measured by the function $\sqrt{x^2 - y^2}$. This, with its attendant consequences, sufficiently defines the locality.

Let AB be a straight line. Numbers give the ordination of positions on it. The length between any two positions is a physical notion. If p is a material rod the intervals AC and BD are said to be equal if the rod p occupies at one time the interval AC, at another time BD without observable distortion in the transference. Taking a two dimensional number system we have besides the system ABC another system of positions $AA'A''$, $BB'B''$ and so on. The

positions A, A' are said to be at the same distance as A, B if the same material rod will occupy the interval A, A' as occupied AB, it being presumed that no distortion takes place.

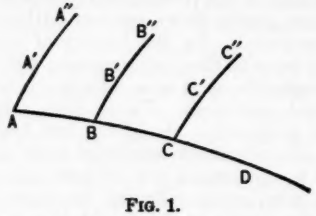

FIG. 1.

The operation of transferring a rod which occupies the interval AB to assume the position AA' is termed rotation. Where in the twofold number system the extremity which is not fixed at A will be found during the process of transference is not given *a priori*. We will suppose a number system so arranged in a material universe that the intervals AB, BC, CD can all in turn be occupied by the same rod, also AA', $A'A''$ and each set, such as BB', $B'B''$, and finally each set, such as $A'B'$, $B'C'$, etc.

So much being settled we are in accordance to a certain extent with our ordinary system of coördinates. But the po-

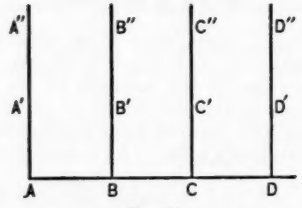

FIG. 2.

sitions which the extremity of the rod will occupy in its rotation have still to be defined. It has in no way been precised by the assumption already made. The shape of the "circle" is as yet indeterminate. Let x and y be the numbers denoting the position of the moveable extremity of the rod in the specified two-fold number system. We find from observation that in our world the law of rotation is this. During rotation $\sqrt{x^2 + y^2}$ has the same value that x alone has before

rotation commenced. Now in Hyperbolea $\sqrt{x^2 - y^2}$ has the same value that x alone has before rotation commenced. From the particular law of length which we assume springs the definition of angle, and all subsequent geometrical relations. Hence the system of rectangular coördinates above drawn is very false if assumed to be a faithful representation of the way of measuring the plane in hyperbolea. But we are familiar with it. It is most practical. In fact it is the only practical way to use our own length and angles in our drawings. Then the physical laws of Hyperbolea will be given as a distortion of our physical laws. There is no help for this in our physical conditions, and we must remember that our physical laws of motion and rotation would seem just as distorted and curious to the Hyperboleans as theirs do to us.

To take a definite instance, let OA, fig. 3, be a rod of length 32 units. It will in Hyperbolea assume positions such that the coördinates of its extremity are $P(40, 24)$ $Q(68, 60)$ $R(130, 126)$. These points lie on the hyperbola $x^2 - y^2 = 32^2$.

Thus if we represent the locus of points in Hyperbolea equally distant from O, we represent it in our physical system as a hyperbola.

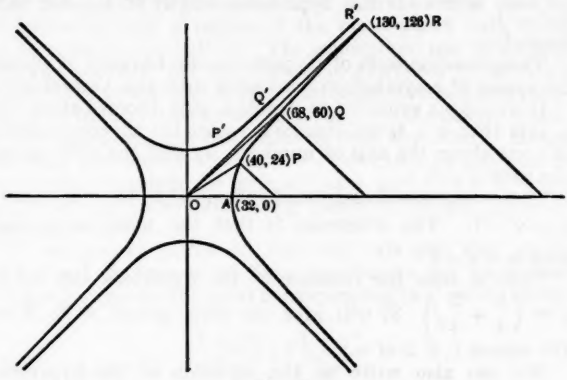

FIG. 3.

OP, *OQ* and *OR* are all equal in length (hyperbolic length).

If $P'Q'R'$ lie on the conjugate hyperbola and are conjugate radii it is evident that if any one such as OP' is to be a real length the coördinates of P' must be imaginary, thus $24\sqrt{-1}, 40\sqrt{-1}$ will be the coördinates for P' and so on

for the other points. Under this assumption $\sqrt{x^2 - y^2}$ will denote a real length. It is evident that a point on the asymptote is at a null distance from the origin. We have in fact in the asymptotes the lines of null length of the hyperboleans drawn extended. Thus we are able to show constructions which would only be matters of symbolism with them. Conversely they can put in extensive visible form constructions which we in our physical system know of merely as algebraical analogies. It is not only from O that the lines of null length start. From every point there are two directions in which the hyperbolean counts length as null. His plane, as ours, is granulated with lines of null length.

With regard to angle, the hyperbolean calls those angles equal which intercept equal arcs on his curve of equal distances. As equal distances are measured by the square root of the difference of the squares of the coördinates, $ds = \sqrt{dx^2 - dy^2}$ and $s = \sqrt{-1}\, a \log (x + y)$. Putting $x = a \cosh u$, $y = a \sinh u$, we get $ds = \sqrt{a^2 \sinh^2 u - a^2 \cosh^2 u}\, du$ or $s = \sqrt{-1}\, au$.

Hence for the hyperbolean the measure of an angle is $\frac{s}{a\sqrt{-1}}$ where s is the hyperbolic length of the arc subtended.

The geometric facts of Hyperbolea can be easily exhibited by means of a symbolism analogous with the Argand affix.

If we take a symbol j to denote a unit directed along the y axis then $a + bj$ will denote a vector having components, the one along the axis of numbers, a; and the other along the axis of y, b.

These vectors will compound and multiply like the vectors $a + b\sqrt{-1}$. The difference is that the modulus in this case is $\sqrt{a^2 - b^2}$.

Thus to take the instance of the hyperbola last taken $\rho = \left(\frac{5}{4} + \frac{3}{4} j\right)^n$ 32 will give the three points P, Q, R for the values 1, 2, 3 of n.

We can also write as the equation of the hyperbola $x^2 - y^2 = 1$,

$$\rho = e^{ju} = 1 + ju + \frac{(ju)^2}{1.2} + \cdots = \cosh u + j \sinh u.$$

This equation gives the properties of the hyperbola. Thus

$$d\rho = je^{ju}\, du = (j \cosh u + \sinh u)\, du$$

showing that the tangent is parallel to the conjugate radius.

Now for real values of u we get only the points on one branch of the hyperbola. To get points on the other branch let $u = u + i\pi$ we get $\rho = e^{(u+i\pi)} = -e^{uj}$.

The points in the conjugate hyperbola are obtained by letting $u = u + \frac{i\pi}{2}$, whence

$$\rho \equiv e^{\left(u+\frac{i\pi}{2}\right)j} = ij\,e^{uj} = i(\cos hu j + \sinh u)$$

the coördinates being imaginary. The hyperbolean can draw imaginary lines.

There is no period for real values of u. In Hyperbolea there is besides rotation a process which we may call Saltation, by which it is possible to pass from one vector to its conjugate. Saltation corresponds to an increase of u by $\frac{1}{2}i\pi$ and all dimaeters make this hyperbolic angle of $\frac{1}{2}i\pi$ with their conjugates.

Equal hyperbolic angles seem to us to grow smaller as their locality recedes from what we call the axis of the hyperbola. But the hyperbolean regards our equal angles as variable as increasing, and after a certain time as disappearing into a region of the plane which only exists as a matter of symbolism. The conception, too, of an axis midway between the asymptotes is unknown to him, all points on the hyperbola being at an equal distance, and there being an infinite rotation to the asymptotes, he has no means of distinguishing any one diameter from any other. Hence working with the symbolism $\rho = e^{ju}$ we can take any pair of conjugate diameters for primary axes and all the formulæ will hold, the coördinates being measured parallel to these conjugate ones.

If we project points on the real axis onto the circle $x^2 + y^2 = 1$ by means of lines parallel to the asymptote $x + iy = 0$ we get for the point corresponding to x' on the circle.

$$-\frac{1+x'^2}{2x'}\ ,\ i\frac{1-x'^2}{2x'}$$

which we cannot draw.

The hyperbolean projecting points on his axis of numbers onto the hyperbola $x^2 - y^2 = 1$ gets real points. It is to these points that I wish to call attention. We have an advantage over him in that we can draw these projecting lines, and so a mechanical process of solving equations which is

symbolical to the hyperbolean can be actually carried out by us.

The point in the hyperbola for which $\rho = \cosh u + j \sinh u$ becomes, when projected on the axis of numbers by a parallel to the asymptote $x + y = 0$, a point at a distance from the origin equal to $\cosh u + \sinh u$. Conversely any point on the axis of numbers can be represented by a point on the hyperbola. The coördinates of the point on the hyperbola corresponding to a point a on the axis of numbers are given by $x + y = a$, $x^2 - y^2 = 1$ and are $\cosh u = \frac{a + \frac{1}{a}}{2}$ $\sinh u = \frac{a - \frac{1}{a}}{2}$

If now we draw from the origin vectors to such representative points on the hyperbola we find that:

1° the addition of distances from the origin corresponds to addition of the vectors by the parallelogram law.

2° the multiplication of the distances, considered as giving a point at a distance on the axis of numbers equal to the product of the distances, corresponds to the addition of the two hyperbolic angles of the representative vectors.

The first remark is justified by the fact that the distances on the axis of numbers are projected on the hyperbola by lines making an angle of 45° with the x axis. The second point follows from the fact that the vector of a point on a hyperbola can be written $\cosh u + j \sinh u$. The product of two such vectors having angles u and u' is evidently $\cosh (u + u') + j \sinh (u + u')$.

Putting $j = 1$ corresponds to finding the distance on the axis of numbers represented by the vector $\cosh u + j \sinh u$.

We have thus the means of carrying out the processes of addition and multiplication by vector addition and hyperbolic rotation.

Take for instance the point 2 on the axis of numbers. To find the corresponding point on the hyperbola put $\cosh u + \sinh u = 2$. We get $\cosh u = \frac{2 + \frac{1}{2}}{2}$ $\sinh u = \frac{2 - \frac{1}{2}}{2}$ or $\cosh u = \frac{5}{4}$ $\sinh u = \frac{3}{4}$. The corresponding vector is $\rho = \frac{5}{4} + j\frac{3}{4}$. The point 4 is similarly $\rho_2 = \frac{34}{16} + j\frac{30}{16}$. These points are shown in fig. 3 if 32 is taken for unity; ρ_1 is then $\frac{40}{32} + j\frac{24}{32}$, ρ_2 is $\frac{68}{32} + j\frac{60}{32}$. The vector sum of these

$\frac{108}{32} + j\frac{84}{32}$ becomes 6 if projected back on the axis of x, *i. e.*, if $j = 1$. Thus, whether we add the vectors and project back or whether we simply add the numbers 4 and 2, we obtain the same result.

The product of these two vectors is $\frac{130}{32} + j\frac{126}{32}$ which projects back on the axis of numbers into 8 (put $j = 1$). And this product vector makes an angle with the axis of x equal to the sum of the factor vectors. Put $\rho_1 = \cosh u_1 + j \sinh u_1 = e^{ju_1}$, we get, if $j = 1$, $e^{u_1} = 2$, or $u_1 = \log 2$; in the same way $u_2 = \log 4$ and their product $e^{ju_1} \times e^{ju_2} = e^{j(u_1+u_2)}$. Thus the hyperbolic angles of the three vectors are respectively log 2, 2 log 2, 3 log 2, the sum of the first two equals the third.

In the following remarks I will not go into the question of the feasibility of constructing a chain of mechanism which shall move vectors along the hyperbola by equal, and multiples of equal, hyperbolic angles. Assuming such a mechanism and also one capable of summing vectors, I will show the process of finding the solution of an equation for real roots, and for mixed real and imaginary roots.

1. Take the quantic $x^2 - 3.5x + 3$ which has the roots 1.5 and 2.

$$\text{Put } x = e^{uj}; \quad 3.5 = e^{1.2528j}; \quad 3 = e^{1.0986j}.$$

This corresponds to the projection shown in the diagram, fig. 4, x being taken as unity and therefore u as null.

The vector ρ^2 representing x^2 lies along OA the vector representing $-3.5x$ lies along OB. OC represents the number 3 and OB the coefficient -3.5.

As x increases in value from 1 the moveable vector ρ^2 starts from OA moving by intervals of $2u$; the vector x or ρ starts from OB moving by intervals of u. The vector sum of ρ^2, ρ and OC gives the value of the quantic.

The directions of rotation and saltation are given by the letters s and m, s denoting a small and m a large hyperbolic angle; thus from s to m the hyperbolic angle increases, from m to s it diminishes.

The curve in fig. 5 gives the locus traced out by the extremity of the compound vector which represents the quantic. The numbers affixed show the values of u at the specified points.

It will be seen that the curve cuts the asymptote $x+y=0$ at the points .4 (written for .40554) and .69 written for .6931 these being log 1.5 and log 2.

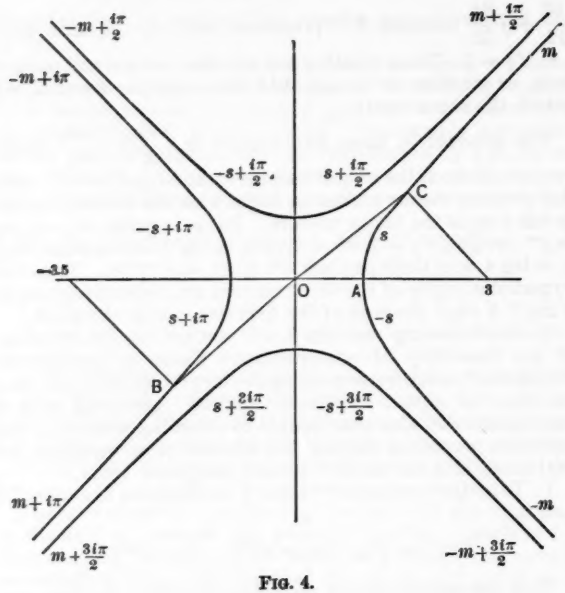
$-m+\frac{i\pi}{2}$
$m+\frac{i\pi}{2}$
m
$-m+i\pi$
$-s+\frac{i\pi}{2}$
$s+\frac{i\pi}{2}$
C
$-s+i\pi$
s
-3.5
O
A
3
$s+i\pi$
$-s$
B
$s+\frac{2i\pi}{2}$
$-s+\frac{3i\pi}{2}$
$m+i\pi$
$-m$
$m+\frac{3i\pi}{2}$
$-m+\frac{3i\pi}{2}$

FIG. 4.

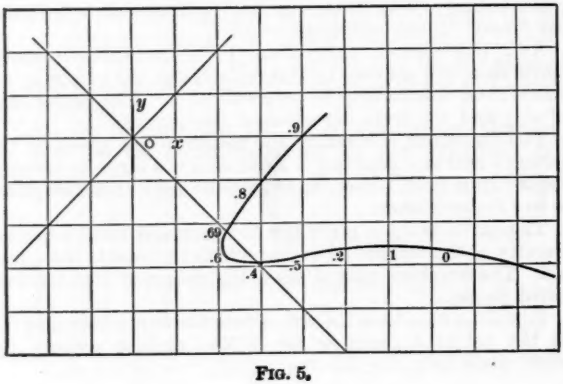
y
O
x
.9
.8
.69
.6
.4
.5
.2
.1
0

FIG. 5.

The projections of these points by a parallel to $x + y = 0$ give a null distance from the origin and hence $e^{\log 2}$ and $e^{\log 1.5}$ are solutions of the equation, namely, 2 and 1.5. Thus, granted the train of mechanism, the solution of an equation (as to its real roots) consists simply in determining the vectors corresponding to the coefficients and then making hyperbolic radii start from them as initial positions at the proper relative rates. The intersections of the curve traced out by the vector sum of the vectors with the asymptote $x + y = 0$ gives values of u for which e^u is a solution.

In default of the mechanism I give the calculation of a couple of the points:

X	$u = .2$	$u = .6931$
$\cosh 2u$	1.0803	2.125
$-\cosh 1.2528 + u$	−2.2534	−3.571
$+\cosh 1.0986$	1.6666	1.666
	+.4935	.220
Y		
$\sinh 2u$	.4105	1.375
$\sinh 1.2528 + u$	2.6201	−3.428
$\cosh 1.0986$	1.3333	1.333
	−.2763	−.220

To exhibit the process of finding both real and imaginary roots add a number to the previous expression so as to make the roots imaginary.

Take $x^2 - 3.5x + 4.625$, the roots of which are $x = 1.75 \pm 1.25\sqrt{-1}$.

As x is to be partly real, partly imaginary, put it equal to $e^{u'} + e^{\left(u' + \frac{i\pi}{2}\right)j}$

That is to say, represent it as a pair of radii the radius having the angle u together with the conjugate of that having the angle u'.

These radii cannot be added together. The single vector sum of them must not be taken. They form an ununifiable couple representing a complex expression. But they can be multiplied together. They form a pair of lines representing the composite expression.

The new equation becomes, substituting the value of x in $x^2 - 3.5x + 4.625 = 0$

$$e^{2u'j} + e^{2\left(u' + \frac{i\pi}{2}\right)j} - e^{(3.5 + u')j} + e^{1.4625j} + 2e^{\left(u + u' + \frac{i\pi}{2}\right)j} - e^{\left(i\,3.5 + u' + \frac{i\pi}{2}\right)j} = 0$$

Here the constant term $e^{1.4625j}$ or $e^{1.5315j}$ differs from the constant term of the preceding equation by $.75 + .87j$. Hence the axes being in the position shown in fig. 6, fig. 5, if its origin be placed at (.75, .87) would represent the value of the left hand side of the equation for real values of u. Fig. 6 is the graph of the new expression. Expanding the exponentials in the expression on the left hand side of the above equation we obtain $\rho = X + Yj + i\,(X' + Y'j)$. Values of X Y etc. are given in the table below at the end of the paper. Taking the vector sum of the real vectors, and of the imaginary vectors we get two series of curves. In default again of the appropriate mechanism the calculation is given below. The mechanical process would differ from that indicated above simply in the circumstance that a means must be provided by which the angles traced out by each of the vectors, the real and the imaginary ones must be introduced so as to add to the rotation of the other in an appropriate way.

In fig. 6 the continuous lines denote the curves traced out by the real vector of the quantic as u and u' vary, the dotted lines denote the curves traced out by the imaginary vector of the quantic as u and u' vary.

Take for instance the curves for which $u = .4$. The real vector sum traces out the hyperbola marked $u = .4$. The second figure in the affixed designations denotes the value of u' for the specified points on the curve.

For the same value of u the imaginary vector sum traces out the hyperbola (dotted) $u = .4$. Corresponding positions on each are shown by the values of u' being identical, *i. e.*, points where both affixed figures are the same, are simultaneous positions of the extremities of the real and imaginary quantic vector.

If u' is constant and u varies the real vector describes hyperbolic sine cosine curves, the imaginary vector describes hyperbolas which all pass through the origin.

The extremity of either vector can move over the whole plane. Thus to any path of the one corresponds a definite path of the other. Hence the method of mechanically obtaining a solution of the equation would be to make the extremity of one vector move along the asymptote, the other then traces a curve which intersects the asymptote. The values of u and u' for these points of intersection give the real and imaginary parts of the root.

In the particular case of the quadratic expression since all the imaginary curves for which $u' =$ constant pass through the origin, the single point of the origin stands for a whole curve of the u series.

This value is in the figure $u' = \log 1.75 = .5597$ (written in the figure .56). Hence in the case of the quadratic the resultant imaginary vector has null components. For

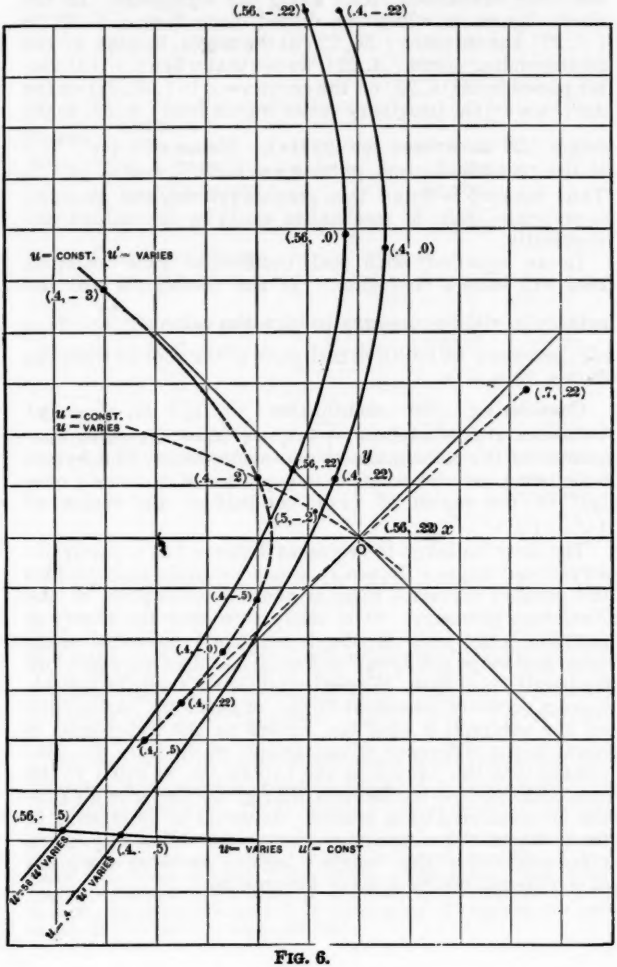

FIG. 6.

a solution the two imaginary vectors can be rotated round so as to form one diameter of the conjugate hyperbola. In general, however, the method would be to move one or the other summation vector along the asymptote. In the above figure, in which the dotted curve from (.4,.22) to (.7,.22) has its point (.56,.22) at the origin, looking at the corresponding points (.4,.22), we see that when the real vector passes from (.4,.22) on the curve $u=.4$ to (.56,.22) on the curve $u=.56$ the imaginary vector travels from (.4,.22) to the origin (.22 is written for .22314). Hence $e^{.559j} + e^{.2231\sqrt{-1}j + \frac{i\pi}{2}}$ is the root which gives, putting $j=1$, $e^{\log 1.75} + \sqrt{-1}e^{\log 1.25}$. Thus the root is found by a graphic process, and given an appropriate train of mechanism could be determined mechanically.

In an equation with real coefficients each complex root will have a conjugate. If the coefficients are imaginary it will be necessary to take the value $e^{u'+\frac{3i\pi}{2}}$. It is also necessary for negative real parts of the root to trace the curves for $u + i\pi$.

Considering the coördinates (x, y) $(x, \sqrt{-1}y)$ $(\sqrt{-1}x, y)$ $(\sqrt{-1}x, \sqrt{-1}y)$, we have access to one-quarter of the plane given by our symbolism. The hyperbolic being has also access to one-quarter of it, namely one-half of the region of $(x\,y)$, one-half of the region of $(\sqrt{-1}x, \sqrt{-1}y)$.

The function taken to represent distance has a purely observational origin. Hence it seems probable that besides the parallel postulate there is another assumption in the Euclidean geometry. It is easily seen that the theory of parallels is the same in the hyperbolean geometry as in ours; and hence it follows that Euclid I.47 does not depend on the parallel postulate. We see that it cannot really be proved. Since a perfectly consistent system of geometry can be built on the assumption that the square on the hypotenuse is equal to the difference of the squares on the sides, the propositon that the square on the hypotenuse is equal to the sum of the squares on the sides of a right angled triangle cannot be considered to be proved. It would be interesting to trace where the assumption comes in. Probably some other property of the "circle" besides its being the locus of equidistant points is taken for granted.

Real curve	$u' = .22314$	
X	$u = .5596$	
$\cosh 2u$		1.6945
$-\cosh 2u'$		−1.1012
$-\cosh (1.2528 + u)$		−3.1441
$+\cosh (1.5316)$		2.4209
		− .1299
Y		
$\sinh 2u$		1.3679
$-\sinh 2u$		− .4612
$-\sinh (1.2528 + u)$		−2.9808
$\sinh (1.53157)$		2.2047
		.1306
Imaginary curve	$u' = .5$	
X'	$u - .4$	
	or	
$\sinh (3.6931 + u + u')$	$+\sinh 1.5931$	2.3591
$-\sinh (1.2523 + u')$	$-\sinh 1.7528$	2.7983
		− .4392
Y'	or	
$\cosh (.6931 + u + u')$	$\cosh 1.5931$	2.5622
$-\cosh (1.2528 + u)$	$-\cosh 1.7528$	−2.9711
		− .4089

LIE'S GEOMETRY OF CONTACT TRANSFORMATIONS.*

SOPHUS LIE.—*Geometrie der Berührungstransformationen.* Dargestellt von SOPHUS LIE und GEORG SCHEFFERS. Erster Band. Mit Figuren im Text. Leipzig, Druck und Verlag von B. G. Teubner. 1896. 8vo. Pp. xi+693.

Lie was asked in conversation once what constitute the necessary and sufficient characteristics of a mathematician. He replied forthwith: "Phantasie, Energie, Selbstvertrauen,

*Lie uses in the German *Berührungstransformation;* his French pupils translate it *transformation de contact*, Forsyth translates it *tangential transformation*, and Klein *contact transformation*. The latter English translation is retained here as a more precise designation, since by such a transformation not only is the property of tangency, but also in general the order of contact preserved.

Selbstkritik." The illustrious Norwegian thereby unconsciously drew an impressionistic portrait of his own genius, as the striking originality and wonderful scope of his mathematical discoveries testify. Lie's phenomenal creative power and indomitable energy are evidenced not only by the numerous memoirs which have appeared from his pen in the journals and proceedings of learned societies of Norway, France and Germany during the last twenty-five years, but even more emphatically by the fact that the above-named volume is the last of six* tomes aggregating over four thousand royal octavo pages that have come from his hand in the interval between 1888 and 1896, the first five of which appeared in the years 1888–'93.

The last one may with all propriety be called Lie's Geometry or the Geometry of Lie's contact transformations. Transformations applied to certain given differential equations had appeared in the science of mathematics as early as Euler, Lagrange and Legendre, but none of these mathematicians studied the transformations in themselves or established general propositions concerning the particular transformations used, much less concerning general categories of such transformations. Ampère studied more general transformations than those of Euler and Lagrange which he applied to partial differential equations, but the idea of a contact transformation appears neither in his writings nor in those of his successors up to the time of Lie's activity. None of Lie's predecessors gave a definition of a contact transformation, much less a systematic theory of contact transformations. To him is due both the credit of inventing the idea of a contact transformation and that of establishing the theory of such transformations in an independent existence. He has also given a precise development of the theory of infinitesimal contact transformations

*SOPHUS LIE: Theorie der Transformationsgruppen. Bearbeitet von Sophus Lie unter Mitwirkung von Prof. Dr. Engel. i. Abschnitt, x+632, 1888; ii. Abschnitt, vi+554, 1890; iii. Abschnitt, xxvii+830, 1893. Leipzig, Teubner.

SOPHUS LIE: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen. Bearbeitet und herausgegeben von Dr. Scheffers, xiv+568, 1891. Leipzig, Teubner.

SOPHUS LIE: Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen. Bearbeitet und herausgegeben von Dr. Scheffers, xii+810, 1893. Leipzig, Teubner.

Vol. 1 of the Theory of Transformation-Groups was reviewed by Dr. Chapman in the BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY, vol. 2, No. 4, January, 1893; the Lectures on Continuous Groups, by Mr. Brooks in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, vol. 1, No. 10, July, 1895.

and groups of contact transformations, these constituting but a part of his more general and beautiful theory of continuous groups. With great injustice to Lie these transformations have been referred to as the contact transformations of Jacobi and Lie. As well, in fact with better propriety could we speak of the method of fluxions of Wallis and Newton or the infinitesimal calculus of Fermat and Leibnitz. All honor to Jacobi for his additions to our knowledge of the means of transforming the so-called canonical systems of partial differential equations in dynamics. It is true that Jacobi used a particular contact transformation. So did Apollonius of Perga two hundred years before the present era. But we have no evidence that either Jacobi or Apollonius knew it, let alone the conception of the most general transformation of this kind; the idea of a contact transformation is Lie's.

To be completed in two volumes the work is intended to give an extended account of Lie's first geometrical investigations* in the years 1869–72. The text of the first volume before us is preceded by a preface of five pages and a table of contents, and followed by an analytical alphabetical index of subjects and also one of the references to more than a hundred mathematicians of the past and present; the marginal references indicating concisely the contents of the paragraphs constitute another marked feature in the way of indices.

The preface is more than an indication of the contents and scope of the sequel. It presents in precise form Lie's ideas relative to the notions that have been of most power in the development of mathematics; his oft-repeated regrets that the science has of late years been broken into two distinct branches—analysis and geometry, to such an extent that one phase almost asserts its independence of the other, and a plea for a kind of reciprocity between analysis and geometry by which they will advance side by side and enrich each other step by step as in the earlier days. Mathematics is broader than geometry, and deeper than analysis, and while Lie is preëminently a geometer, and professionally a professor of geometry,† yet no living mathematician pleads more earnestly for the union of the two great forces in mathematics, and he has taken particular pains to develop

* The best known of these are: "Om en Classe geometriske Transformationer," *Ges. der Wiss. zu Christiania*, October, 1870. Ueber Complexe, insbesondere Linien- und Kugelcomplexe mit Anwendung auf die Theorie partieller Differentialgleichungen," *Math. Ann.*, vol. 5, 1872.

† Professor der Geometrie an der Universität Leipzig.

a rigorous analytical theory* of contact transformations alongside the more obvious geometrical theory.

It is characteristic of Lie's lectures that every advance synthetically is duplicated analytically and *vice versa*. Not the least important result of the dual development of his theory of continuous groups is its wide application to many apparently heterogeneous subjects and hence its great usefulness as a unifying principle in the science. This demand that analysis become the handmaid of geometry and geometry the servant of analysis, was the theme of Lie's inaugural lecture at Leipzig in 1886. How far this theme has been the burden and inspiration of his own mathematical investigations is best shown by the following sentence from the preface, "Charakteristich für meine Richtung dürfte es besonders sein, das ich nach dem Vorbilde von Monge die geometrischen Begriffe, namentlich die von Poncelet und Plücker eingeführten, für die Analysis verwertet und andererseits Lagrange's, Abel's and Galois' Ideen über die Behandlung der algebraischen Gleichungen auf die Geometrie und besonders auf die Theorie der Differentialgleichungen ausgedehnt habe."

The first volume falls into the three grand divisions: I. Contact Transformations in the Plane, chapters 1–5, pp. 1–176; II. Geometry of Lineal Elements in Space, chapters 6–10, pp. 177–480; III. Introduction to the Geometry of Surface Elements. Partial Differential Equations of the First Order, chapters 11–14, pp. 480–687.

I. In the first division of the work is introduced the notion *contact transformation in the plane*, and methods are developed for finding all transformations of this kind. The most important properties of contact transformations are exhibited and their usefulness in geometry and ordinary differential equations illustrated by simple but instructive examples. The infinitesimal contact transformations are defined and all forms of them determined. Applications of these infinitesimal contact transformations to optics and mechanics are given, and finally, their importance in geometry is further shown by a beautiful application in the solution of the difficult problem—to find the form of the elements of arc of all surfaces for which the differential equation of the third order of the geodesic circles† written in

* See vol. 2 of the Theory of Transformation-Groups.

†These so-called geodesic circles are the curves having constant geodesic curvature and were first studied by Minding. They must not be confused with those geodesic circles studied by Gauss under the definition that their points have a constant geodesic distance from fixed points of the

Gaussian coördinates (x, y) admits* of one or more infinitesimal contact transformations of the two-dimensional manifoldness (x, y).

1. As already remarked, long before Lie introduced the general notion of a contact transformation, the notion appeared in the science in special forms. The first chapter is devoted to some of these special forms. It is composed of four sections. The first section is occupied with several historic point transformations† which have played important rôles as correspondences, namely, *orthogonal projection*, the *projective transformation* and *inversion*. Another no less important application of point transformations is their use as auxiliaries in investigations in the theory of invariants; their utility in this direction is shown here by reference to the problem: How can it be determined whether a given ordinary differential equation of the second order in x, y, can be reduced by means of a point transformation to the form $y''=0$?‡ It may be remarked *en passant* that it is always possible to bring an ordinary differential equation of the first order to the form $y' = 0$ by a point transformation; one of the second order to the form $y'' = 0$ by a contact transformation; and a partial differential equation of the first order to the form $z = 0$ by means of a contact transformation properly chosen.

Lie defines a *lineal element* to be the aggregate of a point and a straight line through the point; hence a lineal element is determined by three quantities—the coördinates x, y of the point and the direction y' of the line. A lineal element then has three coördinates x, y, y', and there are accordingly ∞^3 lineal elements in the plane; the plane, a two-

surface. Either property completely defines a circle in the plane, but they define different classes of circles on a curved surface.

* A differential equation is said to admit of a transformation when the family of its integral curves admits of the transformation. A family of curves is said to admit of a transformation when every curve of the family is changed by the transformation into a curve belonging to the same family.

† By a point transformation is meant, geometrically, an operation by which a point is carried into the position of some point. A point transformation of the xy plane into itself is expressed analytically by two equations of the form $x_1 = X(x, y)$, $y_1 = Y(x, y)$, where X, Y are independent analytic functions in the Weierstrassian sense. The so-called *first extension* of this point transformation is given by the equations

$$x_1 = X, \quad y_1 = Y, \quad y_1' = \frac{Y_x + Y_y y'}{X_x + X_y y'}.$$

‡ Lie solved this problem in 1883. It appears that when the solution is possible, for which criteria were established, the reduction depends on the integration of a certain *linear* differential equation of the third order. *Archiv for Math. og Naturv.*, Christiania, 1883, vol. 9, pp. 371 et seq.

dimensional space, thus becomes a three-dimensional manifoldness. A curve determines not only ∞^1 points and ∞^1 tangents, but also ∞^1 lineal elements which are defined to be the lineal elements of the curve. Point transformations are shown to be transformations of the lineal elements of the plane, and an analytical definition of *extended* point transformations is established which is used in the sequel to prove them the simplest forms of contact transformations.

The historic transformations considered up to this point, change point into point and curve into curve; on the other hand there are operations long known which change curve into curve without transforming point into point, such as the *dilatations* which change a curve into its parallel curves, the *pedal transformations* and the *transformations by reciprocal polars*. Sections 2 and 3 are taken up with a discussion of these as transformations of lineal elements and their properties as principles of corrspondencee.

In the fourth and last section the change from point coördinates to line coördinates is shown to be a transformation of lineal elements and the resulting transformation is applied to Clairaut's equation. The section concludes with an application of the transformation by reciprocal polars to the integration of the *generalized* Clairaut equation $y - x\varphi(y') - f(y') = 0$.

2. The second chapter bears the heading "Definition and Determination of the Contact transformations of the Plane." The reader is introduced in the first of the five sections of this chapter to the notion—*element association*.* By an element association Lie means a family of lineal elements (x, y, y') which fulfils the Pfaff equation $dy - y'dx = 0$. He proves that such a family consists of exactly ∞^1 lineal elements; and that points and curves considered as loci of ∞^1 lineal elements are cases of this general notion *element association*. Two neighboring elements (x, y, y'), $(x + dx, y + dy, y' + dy')$ of the family $x = X(t), y = Y(t), y' = P(t)$ are said to be *associated* when $Y' - PX'$ vanishes; or geometrically when to terms of the second order the point of the second lies on the line of the first.

In section 2 there is presented a new conception of the problem of integrating an ordinary differential equation of the first order. To integrate a given ordinary differential equation of the first order in $x, y, F(x, y, y') = 0$ is to find all element associations whose elements satisfy the equation

* Lie uses in the German *Element-Mannigfaltigkeit* and *Elementverein*; the terms *multiplicité* and *association d'éléments* are employed in the French.

$F = 0$.* In this conception an ordinary algebraic equation $F(x, y) = 0$ has integral configurations, namely the points of the curve $F(x, y) = 0$ are general integral configurations while the curve itself is a singular integral configuration of the given algebraic equation. An advantage of this new conception of the problem is that *all* integral configurations of a differential equation are thus made to reappear as integral configurations of the transformed equation when the original is subjected to a transformation; such is not necessarily the case in the classic conception of the integration problem.

The general *contact transformation* in the plane is defined in the third section. Geometrically, a contact transformation is a transformation of the lineal elements which changes an element association into an element association †; analytically, it is a transformation of the variables x, y, y' into the variables x_1, y_1, y_1' by virtue of which there exists an identity of the form

$$dy_1 - y_1' dx_1 = \rho(x, y, y') \ (dy - y'dx).$$

If S and T are contact transformations then the product ST is also a contact transformation. By the *product* of two transformations is meant the transformation equivalent to their successive application. These transformation products do not follow exactly the same laws as ordinary algebraic products; they always obey the associative law but do not of necessity obey the commutative law.

Contact transformations in the plane fall into two classes: 1° *extended* point transformations which change point into point; 2° *proper* contact transformations, which change point into curve or, more strictly speaking, which transform the lineal elements of a point into the lineal elements of a curve. The fourth section is devoted to the determination of all proper contact transformations in the plane by processes of elimination and differentiation with the result that every contact transformation which is not a mere extended point transformation is completely determined by a system of equations

$$\Omega(x, y, x_1, y_1) = 0, \quad \Omega_x + y'\Omega_y = 0, \quad \Omega_{x_1} + y'_1 \Omega_{y_1} = 0,$$

where the equation $\Omega = 0$ is subject to but one condition that the determinant

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* This conception of the problem, and in fact in n dimensions, was given by Lie in the *Göttinger Nachrichten*, October, 1872, p. 481.

† This is the original definition given by Lie in the *Göttinger Nachrichten*, October, 1872, p. 480.

$$\begin{vmatrix} 0 & \Omega_x & \Omega_y \\ \Omega_{x_1} & \Omega_{xx_1} & \Omega_{xy_1} \\ \Omega_{y_1} & \Omega_{xy_1} & \Omega_{yy_1} \end{vmatrix}$$

shall not vanish by virtue of $\Omega = 0$. The theorem is established both synthetically and analytically, and it is observed that the results are wholly independent of the particular system of point coördinates used.

This theorem is applied to numerous examples in the fifth section. Among these are the reciprocal polars, the principle of duality and the dilatations. The forms of contact transformations commutative with certain point transformations are found. Notable among these results are that the dilatations are the only contact transformations commutative with all rotations around the origin and with all translations at the same time; and that those contact transformations commutative with all rotations around the origin and with all uniform extensions from the origin form an infinite group of which the pedal transformations are a particular case.

3. Contact transformations are defined by means of differential equations in Chapter iii. The necessary and sufficient conditions that a transformation of lineal elements

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p)$$

be a contact transformation are found to be

$$[XY] \equiv 0, \quad [PX] \equiv \rho, \quad [PY] \equiv \rho P,^* \quad \rho(x, y, p) \neq 0,$$

where the symbol used since Lagrange and Poisson is introduced, namely,

$$[XY] \equiv X_p(Y_x + pY_y) - Y_p(X_x + pX_y).$$

Two functions X and Y satisfying the condition $[XY] \equiv 0$ are said to be in involution. In the second section of the chapter three interpretations are given to this relation of involution and it is applied to establish categories of differential equations of the first order integrable by elimination.

That ordinary differential equations of the second order have no properties invariant by contact transformations is proved in the third section, *i. e.*, every ordinary differential equation of the second order may be transformed into every other ordinary differential equation of the second order by contact transformations properly chosen. In particular,

* LIE: *Verh. d. Ges. d. Wiss. zu Christiania*, 1872–73; *Math. Ann.*, vol. 8, 1874.

the equation may be transformed into itself, and the condition for this is applied to the equation $y''=0$ to determine the most general contact transformation which changes straight line into straight line. Similar theorems do not obtain for ordinary differential equations of a higher order; in fact, it is remarked that there are differential equations of the third order which do not admit of contact transformations.

The fourth section terminates the chapter with a short historical résumé of points in the writings of Lagrange and Plücker, which were touched incidentally in the developments of the chapter. These historical summaries, of which more occur later, constitute one of the most valuable features of the book to the general mathematical reader.

4. The infinitesimal contact transformations * are powerful implements in the theory of contact transformations and their applications. By an infinitesimal contact tranformation is meant one by virtue of which the lineal element suffers an infinitesimally small change of position. These infinitesimal contact transformations are studied in the fourth chapter. The notion of a one-parameter group of contact transformations stands in close relationship to the notion of an infinitesimal contact transformation. In Lie's terminology a one-parameter group is a family of ∞^1 transformations which possesses the property that the product of any two transformations of the family is a member of the family. Some of the properties of these interesting groups are presented in the first section. Among others that every infinitesimal transformation generates a one-parameter continuous group and conversely, every one-parameter continuous group contains one and but one infinitesimal transformation. These theorems hold in particular when the transformations are contact transformations.

All infinitesimal contact transformations are determined in the second section with the remarkable result that it is possible to give explicitly the general form of an infinitesimal contact transformation in the plane and that this form contains an arbitrary function of the coördinates x, y, p of the lineal elements together with the first derivatives of this function with regard to x, y, p. The result is the more surprising since it is by no means possible to give explicitly the general form of a finite contact transformation, but only to develop methods for their determination. This function $W(x, y, p)$ is called the *characteristic function* of the trans-

* The fundamental notion of an infinitesimal contact transformation Lie introduced in *Verh. der Ges. der Wiss. zu Christiania*, 1872, p. 25.

formation. By virtue of the transformation, x, y, p receive the respective increments:

$$\delta x = W_p\,\delta t, \quad \delta y = (pW_p - W)\delta t, \quad \delta p = -(W_x + pW_y)\,\delta t;$$

and an arbitrary function $f(x, y, p)$ the increment:

$$\delta f = \{[Wf] - Wf_y\}\,\delta t.$$

The last expression divided by δt is adopted as the *symbol* of the infinitesimal contact transformation.

Lie interprets these transformations mechanically as the propagation of a wave motion in an elastic medium and shows that Huygens' principle in optics is a physical expression of the fact that all dilatations form a one-parameter continuous group. Particular categories of infinitesimal contact transformations are examined in this section. Among these are the infinitesimal contact transformations of optics which are defined by the characteristic function $W = \pi(p)$; and those of mechanics for which the characteristic function has the form $W = \Omega(x, y)\sqrt{1+p^2}$. The section is terminated by an account of ordinary differential equations invariant by known infinite simal contact transformations.

Sections 3 and 4 have todo with the differential invariants of infinitesimal contact transformations and their determination. Any function of x, y, p invariant by an infinitesimal contact transformation is defined to be a *differential invariant of the first order*. One of the important theorems developed is to the effect that if one differential invariant of the first order of an infinitesimal contact transformation is known the most general differential invariant of the first order is found by a quadrature. Differential invariants of a higher order and the question of integrating a differential equation of the nth order admitting of a known infinitesimal contact transformation conclude the section.

The fifth and last section deals with commutative infinitesimal contact transformations, and their applications. One beautiful theorem is that two contact transformations $C_1 f$, $C_2 f$ can be brought by new variables introduced by a contact transformation, to the form of translations $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, only in the case when they are independent and commutative. If commutative they satisfy the equation

$$C_1(C_2 f) - C_2(C_1 f) = 0.$$

5. It is the object of the fifth chapter to show the importance of the preceding notions by solving the following difficult problem: To find the square of the element of arc on all surfaces, on which the family of ∞^3 geodesic circles (in Minding's sense) admits of an infinite number of contact transformations. The problem reduces itself at once to that of finding what form the square of the element of arc must possess on those surfaces for which a certain differential equation of the third order admits of one or more infinitesimal contact transformations. We find here an elegant exposition of those beautiful investigations* of Lie relative to geodesic circles which bear a striking analogy to some theorems of Beltrami, Dini and Lie in the theory of geodesic lines. The notation and formulæ of the theory of surfaces are assumed known in this chapter and the first section formulates the problem analytically. The differential equation of the third order of all ∞^3 circles of the surface is reduced to the form

$$w'' - \frac{R}{Z} w + \frac{T}{Z} w^3 = 0,$$

$$\text{where } w = y'^{-\frac{1}{2}}, Z = \frac{1}{\sqrt{z\,(x, y)}}, Z_{xx} = R, Z_{yy} = T.$$

The second section reduces the characteristic function to the form

$$W = 2\,\Omega\,(x, y)\,\sqrt{y'} + \xi\,(x)\,y' - \eta\,(y).$$

Sections 3 and 4 are occupied with the solution of the two cases $\Omega \neq 0$, $\Omega \equiv 0$, respectively. The following are some of the interesting theorems deduced: the family of all ∞^3 geodesic circles of a surface of constant curvature admits of ten independent infinitesimal contact transformations and the family may always be written in the form

$$(x - a)^2 + (y - b)^2 = c^2.$$

If the family admits of only infinitesimal *point* transformations, these are conformal and there are two independent or only one. In the first case

$$ds^2 = \frac{dx\,dy}{[A(x + y)^{\frac{1}{2}+n} + B(x + y)^{\frac{1}{2}-n}]^2},$$

*See Lie, *Archiv for Math. og Naturv.*, vol. 9, 1884, pp. 40 et seq.

and the surfaces are applicable to surfaces of revolution; if in addition the family of ∞^2 geodesic lines of the surface admits of the two independent infinitesimal transformations

$$ds^2 = \frac{dx\,dy}{(x+y)^m}.$$

In the second case that the family admits of but one infinitesimal point transformation the surfaces are applicable to spiral surfaces and ds^2 can be brought to the form

$$ds^2 = \omega(x+y)e^{ax}\,dxdy.$$

In the fifth and last section of the chapter Lie proposes a generalization of ordinary geodesic representation by seeking the cases in which two surfaces may be made to correspond in such a manner that the geodesic circles of the one shall correspond to the geodesic circles of the other. He finds that the representation is conformal and that surfaces between which the correspondence exists are applicable to surfaces of revolution; further that the ds^2 of one determines the ds^2 of its correspondent. We have thus between certain surfaces of revolution a generalization of stereographic projection such that meridian corresponds to meridian and parallel to parallel, while the zones between corresponding parallels are transformed projectively.

II. As in the plane a lineal element in space is defined to be the ensemble of a point and a straight line through the point. The position of the point is determined by its three coördinates x, y, z; and the direction of the line is fixed by the ratios of the increments dx, dy, dz. Hence there are ∞^5 lineal elements in space. A family of ∞^4 will be defined by an equation of the form

$$\Omega(x, y, z; dx, dy, dz) = 0, \qquad (1)$$

which is homogeneous in dx, dy, dz. Lie calls such an equation a Monge equation. If (1) is linear in dx, dy, dz it is called a Pfaff equation and has the form

$$Xdx + Ydy + Zdz = 0. \qquad (2)$$

A curve in space is considered to be the locus of ∞^1 lineal elements. Those curves, the coördinates of whose lineal elements satisfy the equation (1) are defined to be the integral curves of the Monge equation. Two neighboring lineal elements are said to be *associated* when to terms of the

second order the point of one lies on the line of the other. There is a corresponding extension of the notion element association of lineal elements to space which we shall have occasion to refer to later.

6. The mass of matter compressed between the covers of the volume is appalling. Even with the aid of the concise résumés with which each chapter is prefaced any account the reviewer may attempt must be inadequate and unsatisfactory. Chapter vi. is to constitute a connecting link between the first and second grand divisions of the book and bears the heading Pfaff Equations and Nullsystems.* Section 1 interprets the Pfaff equation

$$dy - pdx = 0 \tag{3}$$

in space. Regarding x, y, p, on the one hand, as coördinates of a point in ordinary space, and on the other hand, as coördinates of a lineal element in the xy plane, the simplest form of a Pfaff equation (3) establishes a correspondence between the points of space and the lineal elements of the plane of such a nature that to the integral curves of this Pfaff equation there correspond element associations in the plane. To every contact transformation of the plane there corresponds a point transformation in space which leaves the above Pfaff equation invariant. To integrate an ordinary equation of the first order $F\left(x, y, \frac{dy}{dx}\right) = 0$ is to find the integral curves of the Pfaff equation (3) which lie on the given surface $F(x, y, p) = 0$.

In section 2, Pfaff expressions in three variables are reduced to the canonical forms $d\varphi$, $\Phi d\varphi$, $d\psi + \Phi d\varphi$.

The third section is occupied with nullsystems. The theory of nullsystems or linear line complexes is identical with the study of the special Pfaff equation

$$\begin{aligned} A(ydz - zdy) + B(zdx - xdz) + C(xdy - ydx) \\ + Ddx + Edy + Gdz = 0. \end{aligned} \tag{4}$$

Lie shows that this equation may be transformed into one of the forms

$$xdy - ydx + dz = 0, \tag{5}$$

$$xdy - ydx = 0, \tag{6}$$

* A family of ∞^3 straight lines of which all passing any one point form a flat pencil. *Nullsystem* and *linear line complex* are identical in signification. The family of ∞^3 straight lines cutting a fixed straight line constitute a so-called *special nullsystem*.

by projective transformation. Every general nullsystem admits of a ten-parameter projective group and every special nullsystem an eleven-parameter group of projective transformations. The integral curves of the Pfaff equation (4) are defined as the curves of the nullsystem or linear line complex.*

These complex curves are studied in section 4. Among other remarkable properties all complex curves through a point have the same osculating plane and the same torsion at the point. The latter property is the more remarkable since a nullsystem is defined *projectively* and the notion of torsion is *metric*. The section concludes with a determination of all the asymptotic lines of a ruled surface generated by the lines of the nullsystem and a construction is given by which all the complex curves may be found.

Section 5 establishes a correspondence† between the ∞^3 lines of a nullsystem and the ∞^3 circles in the plane. The equation (5) is reduced to the form (3) and thereby such a transformation of the points of space into the lineal elements of the plane reached that to every straight line of the nullsystem there corresponds a parabola in the plane with axis parallel to a given direction. A certain contact transformation changes these parabolas into circles. Further, by virtue of the equations

$$x_1 = \frac{i(4yz + x)}{8y}, \quad y_1 = \frac{4yz - x}{8y}, \quad p_1 = i\frac{1 - 4y^2}{1 + 4y^2}, \qquad i = \sqrt{-1},$$

the points of space are so transformed into the lineal elements of the plane that to every curve of the nullsystem (5) there corresponds an element association in the plane. In particular to every line of the nullsystem there corresponds a circle in the plane; the ∞^{10} projective transformations which change null-line into null-line are transformed into the ∞^{10} contact transformations which change circle into circle.

7. Lie devotes the seventh chapter to Monge equations and Plücker's line-geometry. The theory of Monge equations is closely connected with that of partial differential equations of the first order. Lie's early researches in geometry showed that Plücker's line geometry falls under the theory of the same equations. These equations are

* LIE: Christiania *Verhandlungen*, 1871, 1883; *Math. Ann.*, vol. 5, 1872, pp. 154, 179.

† LIE: *Math. Ann.* 1872; *Göttinger Nachrichten*, 1874; Transformationsgruppen, vol. 2, chap. 24.

studied directly in the first section of the chapter. A Monge equation (1) as already observed, defines a family of ∞^4 lineal elements in space. Through every point (x, y, z) there pass ∞^1 lineal elements of the family; these form an infinitesimal cone which is defined to be the *elementary cone* of the point. The equation (1) makes correspond to every point an *elementary cone*. An integration problem of (1) then is to find all surfaces which are tangent at all their points to the *elementary cones* corresponding respectively to the points. In case the cone becomes a flat pencil the Monge equation reduces to a Pfaff equation. By introducing the notion *surface element* this problem may be otherwise formulated. A surface element is the ensemble of a point and a plane through the point. There are ∞^3 points in space, and ∞^2 planes through every point; hence there are ∞^5 *surface elements* in ordinary space. A Monge equation defines a family of ∞^4 *surface elements*. The above problem then becomes, to find all surfaces whose *surface elements* belong to this family of ∞^4 surface elements. The problem is solved by the integration of a certain partial differential equation of the first order. An integral curve of the equation (1) is a curve the coördinates of whose lineal elements satisfy the equation. In case ∞^3 straight lines are integral curves of this equation it has the form:

$$\Phi(ydz - zdy,\ zdx - xdz,\ xdy - ydx,\ dx, dy, dz) = 0. \quad (8)$$

Section 2 is a most valuable historical résumé in fine print of earlier investigations relative to families of straight lines in space. With characteristic clearness and conciseness the researches of Euler, Monge, Malus, Binet, Ampère, Giorgini, Möbius, Chasles, Hamilton, Kummer, Transon, Grassmann, Plücker and Cayley are put in proper perspective.

Section 3 gives the fundamental principles of Plücker's geometry, and section 4 treats of the properties of pencils and nets of linear line-complexes and complexes in involution.

Section 5 investigates relations between line-geometry and differential equations. Here we find Lie's extension of the notion of a line complex to that of a *complex of curves; complex curve* and *complex cone* take the place of *complex line* and *elementary cone*. As in the case of nullsystems discussed in the preceding chapter, the theorems relative to the osculating plane and the torsion obtain here also. Lie proves in conclusion that the characteristics of the integral-surfaces are asymptotic lines, and that to find the integral-

surfaces is to find the surfaces on which one family of asymptotic lines is made up of curves of the complex.

8. The eighth chapter bears the heading: "On the Transformation-theory of Tetrahedral Complexes." In this chapter it is proposed to illustrate the close connection between line-geometry and the notions lineal element, Pfaff equation and Monge equation, by a closer study of a remarkable line-complex of the second degree, the so-called *tetrahedral complex*, *i. e.*, the ensemble of all straight lines in space which cut the faces of a tetrahedron in the same anharmonic ratio. This aggregate is composed of ∞^3 straight lines, and since the anharmonic ratio may have ∞^1 values, every tetrahedron has ∞^1 tetrahedral complexes.

The first section contains the fundamental properties of these tetrahedral complexes and an account of Lie's early investigations* concerning them. The Monge equation of a tetrahedral complex has the form

$$(b-c)\,xdydz + (c-a)\,ydzdx + (a-b)\,zdxdy = 0.\dagger \quad (9)$$

The ∞^3 projective transformations which leave the vertices of the tetrahedron invariant are commutative in pairs, form a simply transitive continuous group, and leave also the ∞^1 tetrahedral complexes and their ∞^1 Monge equations invariant.

Section 2 is in fine print and comprises an exhaustive historical sketch of older researches on tetrahedral complexes, the additions to the theory made by Binet, Ampère, Dupin, Chasles, Kummer, von Staudt, de la Gournerie, Plücker, Reye and Lie are reviewed in succession.

Lie's own theory of curves of tetrahedral complexes is presented in the third section. A curve of the complex is one all of whose tangents cut the tetrahedron in the same anharmonic ratio. The tetrahedral complex and its Monge equation (9) admit of the ∞^3 projective transformations

$$x_1 = \lambda x, \quad y_1 = \mu y, \quad z_1 = \nu z. \quad (10)$$

The ∞^3 lineal elements into which a lineal element is transformed by (10) are said to be of the same species.‡ Similarly a curve or surface is changed respectively into a family of ∞^3 curves or ∞^3 surfaces, invariant by the transformation (10); the ∞^3 curves or ∞^3 surfaces are said to be curves or surfaces of the same species. One class of curves

* *Göttinger Nachrichten*, January, 1870.

† It is observed that this equation contains but one parameter.

‡ Gattung.

of the same species is made up of the tetrahedral symmetric* curves; the latter are defined by the equations

$$\begin{aligned} Ax^m + By^m + Cz^m + D &= 0, \\ Ex^m + Fy^m + Gz^m + H &= 0. \end{aligned} \tag{11}$$

Section 4 considers certain transformations of the Monge equation of a tetrahedral complex into itself. All the developments not known before 1870 are Lie's. The Monge equation (9) admits of the transformations

$$x_1 = \lambda x^m, \quad y_1 = \mu y^m, \quad z_1 = \nu z^m; \tag{12}$$

the fact of this invariance leads to a number of important results among which are the determination of the asymptotic lines of the tetrahedral symmetric surfaces, the anharmonic properties of secants of space curves of the third order and of the right line generatrices of surfaces of the second degree, and the fundamental property of Steiner's surface, namely, that every tangent plane cuts the surface in two conics.

In the fifth and last section† Lie introduces his *logarithmic representation:*

$$x_1 = \log x, \quad y_1 = \log y, \quad z_1 = \log z; \tag{13}$$

by virtue of this transformation the ∞^1 Monge equations (9) of a tetrahedral complex are changed into the ∞^1 Monge equations

$$(b - c)\, dy_1 dz_1 + (c - a)\, dz_1 dx_1 + (a - b)\, dx_1 dy_1 = 0, \tag{14}$$

of the line complex of the second degree formed by all secants of a conic belonging to a pencil. Of the many correspondences established by means of this transformation only the following can be mentioned: 1° a plane is transformed into a *surface of translation*‡ which may be generated in ∞^1 ways; 2° a surface of the second degree is changed into a surface of translation which may be generated in four ways.

9. Certain Partial Differential Equations of the Second Order in Line-Geometry. The geometrical problems discussed in this chapter are expressed analytically by partial differ-

*De la Gournerie—Recherches sur les surfaces réglées tétraédrales symétriques. Paris, 1867.

† The developments of this section were worked up by Lie in the winter of 1869–'70.

‡ By a *surface of translation* Lie means a surface containing ∞^1 curves congruent and similarly placed.

ential equations of the second order; they are solved by special methods so that the general theory of such equations is not assumed, and only so much of the theory of partial differential equations of the first order is used as has been developed in the preceding chapters.

The first section solves the problem of determining the surfaces, one of whose families of asymptotic lines belongs to a line complex, yielding the solution that the surfaces sought are integral surfaces of the partial differential equation of the second order

$$r + 2Ns + N^2t = 0, \tag{15}$$

where r, s, t have the usual signification and N is a function of x, y, z, p, q given by the system

$$\Phi = 0, \quad dz - pdx - qdy - 0, \quad dy - Ndx = 0,$$

by eliminating dx, dy, dz; where $\Phi = 0$ is the equation (8) of this review. The theory is illustrated by a number of beautiful examples.

The second section proposes the general problem*—Given a non-linear line-complex; to find all surfaces which contain two families of conjugate curves of the complex. Such surfaces are said to be conjugate with regard to the line-complex. The problem is solved for the particular case when the complex is composed of all the secants of a plane curve. The solution determines among other results all minimal surfaces in space. When the plane curve lies at infinity the conjugate surfaces are the integral surfaces of the partial differential equation of the second order

$$R(p, q)\, r + 2S(p, q)\, s + T(p, q)\, t = 0; \tag{16}$$

in this case the conjugate surfaces are either developable surfaces or surfaces of translation.

The same problem is solved for the tetrahedral complex in the third section; the solution is embodied in the theorem that all surfaces conjugate to a given tetrahedral complex are integral surfaces of the partial differential equation of the second order

$$(b - c)\, xqr - [(b - c)\, xp + (c - a)\, yq + (a - b)\, z]\, s + (c - a)\, ypt = 0. \tag{17}$$

A construction for these surfaces is the *logarithmic* trans-

* The investigations of this section appeared for the first time in a memoir of Lie's in vol. 2 (1877) of the *Archiv for Math. og Naturv.*, Christiania.

formation applied to them, their asymptotic lines are found by quadrature, and examples of algebraic surfaces of this class given, notably Steiner's surface and the so-called *special Plücker complex surface.**

The fourth and last section discusses the relations between the theory of surfaces of translation and Abel's theorem. Lie seeks the surfaces conjugate to two line complexes composed of the secants of a plane curve at infinity and finds that they must satisfy a system of two linear partial differential equations of the second order homogeneous in r, s, t, of the form (17); and further that the conditions of integrability of the system demand that the curve at infinity be an algebraic curve of the fourth order. The connection between Lie's theory of surfaces of translation and Abel's theorem on algebraic integrals of the first species is embodied in the following theorem: If $F(\xi, \eta) = 0$ is the equation of any curve of the fourth order and we put

$$\int \frac{\xi d\xi}{F_\eta} \equiv \Phi(\xi), \quad \int \frac{\eta d\xi}{F_\eta} \equiv \Psi(\xi), \quad \int \frac{d\xi}{F_\eta} \equiv X(\xi),$$

the equations

$$x = \Phi(\xi_1) + \Phi(\xi_2), \quad y = \Psi(\xi_1) + \Psi(\xi_2), \quad z = X(\xi_1) + X(\xi_2)$$

represent a surface generated in four ways by the translation of a curve. Some cases where the curve of the fourth order is reducible are studied in detail and the discussion leads to the surfaces of translation of the preceding chapter, to Scherck's minimal surface, to the minimal helicoid and to Cayley's ruled surface of the third order.†

10. In the introductory paragraph to the tenth chapter Lie characterizes it as the most important one of the volume and states that the investigations of this chapter will be given a wider extension in the second volume. The chapter is headed: Relation between Propositions concerning Straight Lines and Spheres. The first section deals with conformal point transformations in space and the representation of circles in the plane as points in space. This section furnishes the point of departure for the numerous re-

* The surface is $x = \alpha\frac{(a+u)^2}{a+v}, \quad y = \beta\frac{(b+u)^2}{b+v}, \quad z = \gamma\frac{(c+u)^2}{c+v}.$

† The three surfaces last named are respectively

$$e^z = \frac{\sin x}{\sin y}, \quad z = \arctan\frac{y}{x}, \quad y^3 + 2xy - 2z = 0.$$

searches of the chapter which culminate finally in Lie's wonderful contact transformation by which straight lines are changed into spheres. Liouville's theorem is established proving that there are ∞^{10} conformal point transformations in space and that all are one-to-one; further that every such transformation may be represented as the product of similitudinous transformations and transformations by reciprocal radii. The well known transformation studied by Chasles, Möbius, Laguerre, Cayley, and Darboux is then given, namely

$$x = X, \quad y = Y, \quad r = iZ, \tag{18}$$

by which to every point X, Y, Z of space there corresponds a circle

$$(x - X)^2 + (y - Y)^2 + Z^2 = 0$$

of the plane $Z = 0$. By means of this transformation Lie constructs a correspondence between the minimal lines of space and the lineal elements of the plane; the correspondence brings to light many properties of minimal curves and minimal developable surfaces.

The second section investigates the connection between the conformal point transformations of space and the contact transformations in the plane and deduces another interesting property of the transformation (18), namely, that by virtue of (18) to every conformal point transformation of space there corresponds a contact transformation of the plane by which circle is changed into circle.* The section concludes with a determination of all infinitesimal conformal point transformations in ordinary space. The most general infinitesimal conformal point transformation is formed by taking the sum of the following ten independent ones multiplied by constant coefficients:

$$p, q, r; \; yr - zq, zp - xr, xq - yp; \; xp + yq + zr;$$

$$(-x^2 + y^2 + z^2)p - 2xyq - 2xzr, -2yzp + (x^2 - y^2 + z^2)q$$

$$- 2yzr, -2zxp - 2zyq + (x^2 + y^2 - z^2)r;$$

where p, q, r, are written for $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ respectively. The first six constitute the group of Euclidean motions and the

* Lie, *Göttinger Nachrichten*, May, 1871, p. 201; *Math. Ann.*, vol. 5, 1872, p. 186.

first seven the group of similitudinous transformations.

Lie examines the relations between the linear line complex and the complex of all minimal lines in section 3 and applies and extends some of the transformations considered in chapter VI. If we regard s, F, F' as the coördinates of a minimal line whose equations are

$$\begin{aligned}(1-s^2)\,X+i(1+s^2)\,Y+2sZ+F&=0,\\ 2sX-2isY-2Z-F'&=0,\end{aligned}\tag{19}$$

we may verify that every conformal transformation of space transforms these coördinates in such a manner as to leave the Pfaff equation

$$dF-F'ds=0\tag{20}$$

invariant; and that the minimal lines passing through a point are given by a relation of the form

$$F=a+2bs+cs^2.$$

If s, F, F' are now interpreted as the coördinates of a lineal element of another plane, the conformal transformations of space will correspond to the contact transformations of the plane which leave invariant the family of all parabolas having parallel axes; whence the transformation of the sixth chapter which changes these parabolas into circles. Lie combines this correspondence between the minimal lines of space and the lineal elements (s, F, F') of the plane with the transformation of chapter VI., which changes the lineal elements of the plane into the points of space. He thus obtains a one-to-one correspondence, defined by the equations

$$\begin{aligned}X+iY+xZ+z&=0,\\ x(X-iY)-Z-y&=0,\end{aligned}\tag{21}$$

between the lineal elements of the Pfaff equation

$$xdy-ydx+dz=0\tag{5}$$

and those of the Monge equation

$$dX^2+dY^2+dZ^2=0.\tag{22}$$

The points of the space (x, y, z) are changed into the minimal lines of the space (X, Y, Z) and the points of the space (X, Y, Z) are transformed into the lines of the linear complex (5).

This transformation is studied further in the fourth section which has for its subject a correspondence between the straight lines of one space and the spheres of another. Lie finds that, by virtue of the transformation (21), the straight lines of a ray system of the first order and first class, that belongs to the linear complex (5), *i. e.*, the straight lines of the space (x, y, z) cutting two reciprocal polars:

$$x = rz + \rho,\ y = sz + \sigma;\ x = -\frac{r}{\eta}z - \frac{\rho}{\eta},\ y = -\frac{s}{\eta}z - \frac{\sigma}{\eta},$$

$$(\eta \equiv s\rho - r\sigma),$$

correspond to the points (X, Y, Z) of a sphere in the space (X, Y, Z)

$$\left(X - \frac{\rho + s}{2r}\right)^2 + \left(Y + i\frac{\rho - s}{2r}\right)^2 + \left(Z - \frac{\eta - 1}{2r}\right)^2 = \left(\frac{\eta + 1}{2r}\right)^2.$$

The equations (21) determine a contact transformation of the surface elements of space by which the surface elements of a straight line and of its reciprocal polar correspond to the surface elements of a sphere. The most beautiful property of this transformation of Lie is that it changes the asymptotic lines of a surface into the lines of curvature of its transformed surface.

The last section of the chapter is devoted to associations of lineal elements in space. As in the plane, two lineal elements in space are said to be *associated* when to terms of the second order the line of the first element contains the point of the second. An *association* of lineal elements is a family of elements in which every element of the family is *associated* with all infinitesimally neighboring elements of the family. An *association* of lineal elements in space consists of the ∞^1 elements of a curve, the ∞^1 elements of an elementary cone, or the ∞^2 elements of a point; there are no other varieties. The theorem that the point transformations are the only ones by which every element association of lineal elements in space is transformed into an element association concludes the section.

III. It has already been observed that the third grand division of the volume contains an introduction to the geometry of surface elements and the theory of partial differential equations as a part of that geometry. The family of all ∞^5 surface elements of ordinary space is conceived to be a five dimensional manifoldness and the quantities x, y, z, p, q, are coördinates for the determination of an element of this

manifoldness. One or more relations in x, y, z, p, q determine families of surface elements. If there is but one relation it takes the form of a partial differential equation of the first order:

$$F(x, y, z, p, q) = 0;$$

if this relation is free from p and q, it determines a family of ∞^4 surface elements whose points lie on the given surface

$$F(x, y, z) = 0$$

and whose planes are arbitrary. Two relations in x, y, z, p, q give a family of ∞^3 surface elements. This family may consist of the elements of the ∞^1 integral surfaces of a certain Pfaff equation, or of those whose points lie on a certain surface, or of those whose points generate a certain space curve. Families of ∞^2 surface elements will be given by three relations in x, y, z, p, q. Among the number of possibilities in this case the family of ∞^2 elements may consist of those of a surface, or of those of a curve or of those of a point; accordingly surfaces, curves and points appear as special forms of two dimensional manifoldnesses in the five dimensional manifoldness of all surface elements.

The geometry of surface elements is an analogue of Plücker's line geometry. Plücker regarded the family of all ∞^4 straight lines of ordinary space

$$x = rz + \rho, \quad y = sz + \sigma$$

as a four dimensional manifoldness, in which r, ρ, s, σ determine the elements, namely, the straight lines. According as there are given one, two or three relations in r, ρ, s, σ we have nullsystems, raysystems or ruled surfaces. Just as the *projective* transformations play a leading rôle in the *line geometry* so the *contact transformations* come to the front in the *geometry of surface elements*. This geometry of surface elements was founded by Lie in his first geometrical work in the year 1870. The last division of the first volume is intended to give an introduction to this geometry only so far as it relates to partial differential equations of the first order; *i. e.*, to the theory of families of ∞^4 surface elements. This geometry will be further developed in the second volume. In the first chapter of the division Lagrange's theory is given, in the second the theory from the standpoint of Lie's geometry, while the last two chapters are concerned with special categories of partial differential equations of the first order.

11. Lagrange's Theory of Partial Differential Equations of the First Order and its Geometrical Interpretation according to Monge. These classic theories are here reproduced in a concise and elegant form. The first section has do with linear partial differential equations and systems of *simultaneous* ordinary differential equations. Among other things, it contains a generalization for a system of $n-1$ equations in n variables and introduces the notion *integral manifoldness*.

The second section gives Lagrange's development of a general solution of a partial differential equation from a complete solution. Lie renders the classic treatment of this problem particularly simple and satisfactory by making continual use of the notion surface element. The general case is illustrated by a number of examples happily chosen.

The notion of the complete integral and the generation of integral surfaces by the method of envelopes leads to the definition of the *characteristics* and their fundamental properties given in the third section, which studies the generation of integral surfaces by the characteristics. The term characteristic is used here in the sense introduced by Monge. A non-linear partial differential equation of the first order has ∞^3 characteristics. Through every characteristic there passes a family of an infinite number of integral surfaces. For this reason the characteristics are called *curves of indetermination*.*

This observation that, when the integral surfaces are to be determined, the characteristics are curves of indetermination, leads to the differential equations of the characteristics in the fourth section of the chapter. Sections 3 and 4 are illustrated by examples.

The fifth section ends the chapter with an extended historical recapitulation of the early investigations in the theory of partial differential equations of the first order. The contributions of Euler, d'Alembert, Lagrange, Jacobi, Lacroix, Charpit, Monge, Bonnet, Lie, du Bois-Reymond, Cauchy, Grassmann, Cayley and Riemann are successively reviewed with the conclusion that Lie was the first to apply the notions of the theory of manifoldnesses to the theory of partial differential equations.

12. The Theory of Partial Differential Equations of the First Order as a Part of the Geometry of Surface Elements. The notion *element association* of surface elements is intro-

* Unbestimmtheitscurven.

duced and a new formulation of the integration problem presented in the first section. By an *element association* of surface elements Lie means a family of surface elements the coördinates x, y, z, p, q of whose elements satisfy the Pfaff equation

$$dz - pdx - qdy = 0.$$

Such a family consists of ∞^2 or ∞^1 surface elements; in the former case the *element association* is composed of the surface elements of a *point, curve* or *surface*; in the latter, either of those of an *elementary* cone, or of those of an *element band*,* where by an *element band* is meant such a family of ∞^1 surface elements along a curve that the points of the elements are points of the curve and the planes of the elements are tangent planes to the curve at the respective points. To integrate a partial differential equation of the first order

$$F(x, y, z, p, q) = 0$$

is to find all element associations whose elements belong to the family of ∞^4 elements defined by the equation. This generalized conception of the problem throws light on the meaning of the terms *solution, general solution, complete solution* and *singular solution*. By introducing his notion *characteristic bands* in place of Monge's *characteristic curves* Lie has also generalized the notion of a complete integral.

The second section is devoted to Lie's *characteristic bands* and their representation as lineal elements in the plane. These characteristic bands are defined in a manner analogous to the definition of Monge's characteristics by assuming the existence of a complete integral. Two correspondences are derived in this section by each of which the *characteristic bands* are represented as lineal elements in the plane and the integrals correspond to element associations of lineal elements in the plane. These transformations differ in that by one the integral configurations of a certain complete solution are represented as points, and by the other as ∞^2 curves in the plane. The correspondence between the two planes of these correspondences is effected by a contact transformation. The derivation of the ordinary differential equations of the characteristic bands terminates the section.

The third section presents the proof of the existence of a complete solution. Starting from the differential equations of the characteristic bands Lie demonstrates by a rigorous process which avoids the famous objection made to the

* Elementstreifen.

method of Cauchy, that the integration of this system of ordinary differential equations leads to a possible construction for a complete solution of the given partial differential equation of the first order. In what has gone before Lie has developed his theory of characteristic bands as a parellel to Monge's theory of characteristic curves; he concludes this section, however, with a *direct* theory of partial differential equations in terms of his notions *element associations* and *characteristic bands*, independent of the original development.

The relation of involution already introduced in chapter III is studied further in section 4 of this chapter. Two partial differential equations of the first order have ∞^1 integral configurations in common when the two are in involution. The problem of integrating the ∞^1 partial differential equations of the first order

$$F(x, y, z, p, q) = \text{constant},$$

is solved as soon as two independent functions Φ and Ψ of x, y, z, p, q are known which are in involution with each other and with F.

The fifth section ends the chapter with a discussion of transformations of partial differential equations of the first order. By a point transformation the characteristic bands of an equation are changed into the characteristic bands of the transformed equation. The relation of involution is invariant by a point transformation. The representation of the characteristic bands as lineal elements in the plane leads to the beautiful result that there corresponds a contact transformation of the plane to every point transformation of space which leaves invariant the partial differential equation considered.

13. The thirteenth chapter treats of *partial differential equations of the first order which admit of infinitesimal point transformations.* The investigations of this chapter touch in many points Lie's theory of continuous groups and its applications to differential equations, but it is not his purpose to go into the general theory of groups in this work and accordingly he gives only some of his early (1870–71) reseaches in this direction as a preparation for the more general theories and as an indication of their importance to geometry.

The first section deals with categories of equations which admit of infinitesimal translations or rotations. If a partial differential equation of the first order admits of two independent infinitesimal translations $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, it is free

from x and $\dot{y}$, in involution with the equation $\frac{p}{q} = a$, and the complete solution is given by a quadrature. If it admits of an infinitesimal rotation about the z-axis it is in involution with the equation $xq - yp = 0$. Other beautiful theorems are deduced for infinitesimal screws and infinitesimal projective transformations.

In section 2 Lie finds classes of equations admitting of a known infinitesimal transformation. But a few of the novel theorems deduced may be mentioned: 1° partial differential equations of the first order admitting of an infinitesimal rotation around the z-axis have the general form:

$$\Omega(x^2 + y^2, z, p^2 + q^2, px + qy) = 0;$$

2° if a partial differential equation of the first order admits of

$$Uf \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}, \tag{23}$$

it is in involution with the linear equation

$$\xi p + \eta q - \zeta = 0;$$

3° the most general line complex admitting of an infinitesimal rotation around the z-axis is

$$\Psi(r^2 + s^2, \rho^2 + \sigma^2, s\rho - r\sigma) = 0.$$

Section 3 is taken up with equations which admit of two commutative infinitesimal transformations. If a non-linear partial differential equation of the first order admits of two independent commutative infinitesimal transformations it is integrable by a quadrature.

The fourth and last section of the chapter has to do with equations admitting of two non-commutative infinitesimal transformations. Let a given non-linear partial differential equation of the first order, $F = 0$, admit of the two independent non-commutative infinitesimal point transformations $U_1 f$ and $U_2 f$, where the Uf's have the form (23); the equation $F = 0$ is then in involution with

$$J \equiv \frac{\xi_2 p + \eta_2 q - \zeta_2}{\xi_1 p + \eta_1 q - \zeta_1} = \text{constant},$$

and $U_1' J$, $U_2' J$ are integrals of the system of simultaneous ordinary differential equations associated with $F = 0$, where $U'f$ is the *extension* of Uf which includes the increments of

p and q. There are four cases to consider: 1° $U_1' J = 0$; 2° $U_1' J =$ constant; 3° $U_1' J = \varphi(J)$; 4° $U' J$ independent of J. In the first case $U_1 f$ and $U_2 f$ are commutative; in the second $U_1(U_2 f) - U_2(U_1 f) \equiv U_1 f$; in the third, $U_1(U_2 f) - U_2(U_1 f) \equiv a\, U_1 f + b\, U_2 f$, where a and b are constants. In the first and fourth cases the original equation $F = 0$ is integrable by a quadrature, and in the second and third its integration is reduced to that of an ordinary differential equation of the first order and further reduction is impossible.

14. This, the last chapter of the book, discusses *certain classes of partial differential equations of the first order* which are of particular interest in the *geometry of curves on surfaces*. The first class of these are those whose characteristics are asymptotic lines on the integral surfaces; these occupy section 1. For this case $F = 0$ is subject to the condition that the equation

$$(F_z + F_x p)\, F_p + (F_y + F_x q)\, F_q = 0$$

shall exist as a consequence of $F = 0$. All partial differential equations of this class may be found by processes of differentiation and elimination.

The second section is devoted to the class whose characteristics are lines of curvature. Since Lie's transformation of straight lines into spheres changes the asymptotic lines of a surface into the lines of curvature of the transformed surface, the solution of this second problem may be obtained from the solution of the first above by an application of this transformation; for this reason the fundamental properties of contact transformations and of systems and complexes of spheres are presented in this section. Among other interesting details of the exposition are two tables of correspondences between configurations, problems and theorems, established by Lie's line sphere transformation. It appears that the characteristics are lines of curvature only in the case when $F = 0$ satisfies the condition

$$\begin{aligned}&(F_y + F_z q)\, F_p - (F_z + F_z p)\, F_q \\ &+ (F_y p - F_x q + F_z pq)\, (F_p p + F_q q) = 0.\end{aligned}$$

In case the characteristics are asymptotic lines and lines of curvature at the same time the equation either has the form

$$F(p, q, z - xp - yq) = 0,$$

or it is linear and its characteristics are ∞^2 minimal right lines.

The third section seeks those equations whose characteristics are geodesic lines on the integral surfaces. A large class of these is very ingeniously deduced from the preceding by making use of the properties of evolutes* of surfaces; this class has the form

$$(p\,H_x+qH_y-H_z)^2-(H_x^2+H_y^2+H_z^2-1)(p^2+q^2+1)=0,$$

where H is an arbitrary function of x, y, z; Lie gives also a second determination of this same class by means of considerations based on his sphere geometry. For the solution of the general problem of this type $F=0$ is subject to a condition containing second derivatives, namely, that the determinant

$$\begin{vmatrix} p & q & -1 \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \\ \frac{d^2x}{dt^2} & \frac{d^2y}{dt^2} & \frac{d^2z}{dt^2} \end{vmatrix} = 0$$

in consequence of $F=0$.

The fourth and last section considers further categories of partial differential equations of the first order. The solution of the problem of determining all surfaces whose normals belong to a given line complex leads to the integration of an equation of the form

$$F(p, q, x+zp, y+zq)=0.$$

These surfaces were first studied by Transon in 1861; Lie showed in 1871 that the solution of this normal problem finds its analytical expression in those partial differential equations of the first order which admit of an infinitesimal dilatation. The solution of the problem includes also the determination of the surfaces of which ∞^1 geodesic lines are curves of a given line complex. The section concludes the book with a résumé of the problems treated in this last chapter together with indications of other problems formed by combinations of the preceding ones, discussions of which are promised for the second volume.

Relative to the part taken by Dr. Scheffers in the preparation of this volume we can heartily emphasize the words of Lie in the preface: "Ich bin besonders glücklich darüber, dass Herr Scheffers dazu bereit war, mit mir zusammen meine Geometrie der Berührungstransformationen zu redigieren."

* Centraflächen.

As regards both the arrangement and the presentation of subject matter this is the best German text-book and treatise combined that has come to the writer's notice. The mechanical execution is excellent. The type is large and well spaced and leaded. The numerous figures are carefully drawn and the engravings are clear cut. It is a credit to the Teubner press.

The book is delightful reading both for the student of mathematics and for the general mathematical reader who is interested in mathematics but not making it a specialty. It is broadening, stimulating and suggestive; students will find suggestions for many an original note and teachers will pick up many a hint for class work with their more intelligent pupils. Previous acquaintance with Lie's theories is not necessary to an intelligent perusal of the book. Lie's marvelous power of concrete representation is here at its best and mathematicians will look forward with keen interest to the concluding volume of this treatise.

EDGAR ODELL LOVETT.

BALTIMORE,
March 31, 1897.

SHORTER NOTICES.

Plane and Solid Analytic Geometry. By FREDERICK H. BAILEY and FREDERICK S. WOODS, Assistant Professors of Mathematics in the Massachusetts Institute of Technology. Boston and London. Ginn & Co. 1897. 8vo, xii + 371 pp.

ANALYTIC Geometry forms the basis of all higher work in mathematics and requires as preparation, beyond the algebra and geometry required for admission to most colleges and technical schools, nothing except such a knowledge of plane trigonometry as may easily be gained in the course of two or three months. It should therefore in the opinion of the reviewer, form the substantial portion of the mathematical course of the first year of colleges and technical schools. But even if postponed to a slightly later period the subject should, we believe, be treated in a thoroughly elementary manner, *i. e.*, without the use of determinants, abridged notation, cross ratio, the line at infinity, etc. The methods of elementary analytic geometry are not elegant and they are sometimes long, so that it is easy to understand how a teacher of this, as of other elementary subjects, listening perhaps to the advice of some misguided enthusiast may desire to "modernize" the teaching of the subject by discarding the old, cumbrous, but direct for the new, elegant and refined methods. He forgets, however, in doing this that his first duty to the student is to use not the methods which in themselves are the best, but those from which the student will gain most; and in a subject as novel to the beginner as is analytic geometry we are confident that these methods are the ones which will least withdraw the student's attention from the fundamental ideas. Messrs. Bailey and Woods have produced a text-book which is in the main in harmony with the ideas just stated, and which is calculated to give to the student such a grasp of analytic geometry as will enable him to deal with elementary problems with ease and accuracy.

Although, as has been indicated, the authors are on the whole conservative, their book presents several new features. The most striking of these consists in the introduction, as soon as the conic sections have been defined by the method of Boscovich and their shapes discussed, of a treatment of the general equation of the second degree in which the xy term is wanting. We do not think that this treatment will be found difficult even by the dull student and its introduction at this point has the advantage of making

it possible to dispose of the question of tangents, normals and polars for the ellipse, hyperbola and parabola at one stroke.

The small number of formulæ given and of theorems proven is an excellent feature of the book, for the student should be made to look upon analytic geometry not as a mass of formulæ and theorems, but as a method by which he can deduce formulæ and theorems when he wants them. Exercises occupy, as they should, a prominent place, though we regret that such a large proportion of them are of the trivial numerical sort.

Perhaps the most important class of exercises for developing the thinking power of the student are locus problems. Exercises of this sort are not wanting; but little attention is given in the text to the methods of solving them, even the introduction of auxiliary variables and their subsequent elimination not being explicitly mentioned. This is a serious omission, which, however, can be supplied by the teacher.

The portion of the book devoted to plane analytic geometry includes a discussion of some of the more elementary properties of the ellipse, hyperbola and parabola and closes with a chapter on the general equation of the second degree.

Nearly one quarter of the book is devoted to solid analytic geometry and in this space it has been possible to give an introduction to the subject amply sufficient as a preparation for higher mathematical work. At the close the questions of the rectilinear generators and circular sections of quadric surfaces are touched upon.

In conclusion the book deserves praise not only for clearness of statement but in the main for rigor of treatment.

MAXIME BÔCHER.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, on Saturday afternoon, May 29, at three o'clock. Professor F. MORLEY and later the Vice-President, Professor R. S. WOODWARD, occupied the chair. There were eight members present. The Council announced the election of the following persons to membership in the Society: Mr. WILLIAM ANTHONY GRANVILLE, Yale University, New Haven, Conn.; Dr. KURT LAVES, University of Chicago, Chicago, Ill.; Professor ANNA HEL-

ENE PALMIÉ, Western Reserve University, Cleveland, Ohio. Three nominations for membership were received. On recommendation of the Council, By-Law V was so amended that the regular meetings of the Society shall hereafter be held at half past ten o'clock, A. M., on the last Saturday of February, April and October, unless the Council shall otherwise order. The following papers were read:

(1) Professor F. MORLEY: "On the generating function of permutations when considered with reference to sequences."

(2) Dr. EMORY MCCLINTOCK: "On a solution of the biquadratic which combines the methods of Descartes and Euler."

(3) Dr. E. O. LOVETT: "On certain classes of point transformations in the plane."

In the absence of the authors, Dr. McClintock's paper was read by Professor T. S. Fiske, Dr. Lovett's by the Secretary.

THE second conference of members of the AMERICAN MATHEMATICAL SOCIETY was held in Chicago Saturday morning, April 24, at ten o'clock. There were thirteen members present. Professor E. H. MOORE was elected temporary chairman. Under authorization by the Council, the conference proceeded to organize as a Section of the Society. The following officers were elected: Chairman, Professor E. H. MOORE; Secretary, Professor THOMAS F. HOLGATE; Programme Committee, Professor ALEXANDER ZIWET, Professor A. S. HATHAWAY, and the Secretary. The following papers were read:

(1) Dr. E. M. BLAKE: "Note upon a representation in space of the ellipses drawn by an ellipsograph."

(2) Dr. JAMES W. GLOVER: "Some properties of function defined by a certain differential equation."

(3) Professor A. S. HATHAWAY: "Orthogonal surfaces."

(4) Professor A. S. HATHAWAY: "Steady motion of fluids."

(5) Professor E. H. MOORE: "The decomposition of modular systems of rank n in n variables."

(6) Professor H. B. NEWSON: "On canonical binary forms."

(7) Professor JAMES BYRNIE SHAW: "A theory of continuity."

(8) Professor JAMES BYRNIE SHAW: "A general theorem in matrices."

(9) Professor H. S. WHITE: "On matrices, covariants and affiliants."

(10) Professor A. S. HATHAWAY: "Quaternions as four-dimensional numbers."

The next meeting of the Section will be held during the Christmas holidays, the exact date to be fixed hereafter.

AT the meeting of the Council of the AMERICAN MATHEMATICAL SOCIETY, held May 29, Professor E. H. MOORE was elected Second Vice-President of the Society.

DURING the season 1897–98 the AMERICAN MATHEMATICAL SOCIETY will hold four meetings in New York, viz.: on Saturday, October 30; Wednesday, December 29; Saturday, February 26; Saturday, April 30. Each meeting will begin at half past ten o'clock in the morning and will include a morning and an afternoon session. The meeting of December 29 will be the annual meeting.

THE official report of the proceedings of the last annual meeting of the German Mathematical Society, held at Frankfort-on-the-Main, September 21–26, 1896, gives the following list of papers read at this meeting:

(1) A. BRILL (Tübingen): "The resolution of a ternary form into linear factors."

(2) R. FRICKE (Braunschweig): "On a simple group of 360 operations."

(3) F. KLEIN (Göttingen): "On a proposition in the theory of finite (discontinuous) groups of linear substitutions of any number of variables."

(4) G. KOHN (Vienna): "On a geometric interpretation of the invariants of doubly binary forms."

(5) G. LANDSBERG (Heidelberg): "On a particular kind of space transformations."

(6) F. MEYER (Clausthal): "On full systems in trigonometry."

(7) R. HAUSSNER (Würzburg): "On Goldbach's law."

(8) L. HEFFTER (Giessen): "On adjoining configurations (Nachbarconfigurationen), triple systems, and metacyclic groups."

(9) M. NOETHER (Erlangen): "Continuous groups of Cremona transformations."

(10) F. ROGEL (Barmen): "On the multiplicity of values of trigonometric expansions within certain limits."

(11) H. SCHAPIRA (Heidelberg): "On a *cribrum algebraicum*, or the cofunctional generation of prime numbers."

(12) F. SCHILLING (Aachen): "On circular-arc-triangles with simple node."

(13) A. SCHOENFLIES (Göttingen): "Transfinite numbers, the axiom of Archimedes, and projective geometry."

(14) E. STUDY (Bonn): "The problem of Apollonius."
(15) E. SCHRÖDER (Karlsruhe): "On some of G. Cantor's theorems."
(16) J. HAGEN (Washington): "Report on a new bibliography of the works of Leonhard Euler."
(17) K. ROHN (Dresden): "Determination of the number of constants for tortuous curves."
(18) E. STEINITZ: "Homogeneous congruences."
(19) J. FRANZ (Königsberg): "Linear differential equations with an absolute term."
(20) F. KLEIN (Göttingen): "On the analytic representation of rotations in the problems of mechanics."
(21) L. HENNEBERG (Darmstadt): "On hydrostatics."
(22) F. MEYER (Clausthal): "On force effects in triple engines."
(23) B. SCHWALBE (Berlin): "On the preparation required of teachers of mathematics and natural science in higher educational institutions, according to the demands of the present day."
(24) W. DYCK (Munich): "On the resolutions adopted by the international catalogue conference in London, July, 1896."
(25) K. HEUN (Berlin): "On the mathematical and mechanical principles in their application to technical problems."
(26) F. S. ARCHENHOLD (Berlin): "Photographs of the large telescope (70 cm. aperture, 21 m. focal length) of the Treptow Observatory."
(27) H. BURKHARDT (Göttingen): "On vector analysis."
(28) ISRAEL-HOLTZWART (Frankfort): "Proposition for extending the intuitive methods of mathematical representation."
(29) F. HÖFFLER (Zürich): "On a method for determining simultaneously the velocity of light and the velocity of the solar system in space."
(30) G. MIE (Karlsruhe): "On the transmission of energy in the electro-magnetic field."
(31) W. A. NIPPOLDT (Frankfort): "Propositions for improving the isochronism of pendulums of clocks by various simple compensations."
(32) O. RAUSENBERGER (Frankfort): "The discontinuities of fluid motion."
(33) J. R. SCHÜTZ (Göttingen): "Solution of the boundary problem (Randwertaufgabe) for the diffraction of Röntgen rays.
(34) E. WIECHERT (Königsberg): "On the distribution of mass in the interior of the earth."

(35) H. WIENER (Darmstadt): Exhibition of models in the mathematical museum of the Technical School at Darmstadt.

As this meeting of the German Mathematical Society was held, as customary, in connection with the meeting of the German Association of Physicians and Naturalists, the last thirteen of the above-named papers were read at joint sessions arranged with the sections of this Association for physics, for theory of instruments, and for the teaching of mathematics and natural science. The publication of the annual volume (Jahresbericht) of the Society has passed from G. Reimer, of Berlin, to B. G. Teubner, of Leipsic. The fifth volume, which is in preparation, will contain in full the papers of Dr. Schwalbe (23) and Professor Burkhardt (27), as well as abstracts of the other papers.

As regards the publication of more extensive reports on special mathematical subjects, it is announced that, while Professor Hilbert's Report on the theory of algebraic corpora has been published in the fourth annual volume, a report on the theory of numbers by Professor Minkowski will appear in a future volume. Professor E. Czuber, of Vienna, promises a report on the calculus of probabilities for the next annual meeting which will take place at Braunschweig, in September, 1897; Professor P. Stäckel hopes to complete his report on differential geometry within two years. There are also in preparation a report on line geometry, by Professor Wälsh, one on infinite series, by Professor Pringsheim, and one by Professor Mehmke, on graphical methods.

Instead of the mathematical dictionary discussed at previous meetings, it has been decided to publish an Encyclopædia of the Mathematical Sciences; Professors Meyer (Clausthal) and Burkhardt (Zürich) are the editors; B. G. Teubner is the publisher. For the year 1897, the executive committee elected Professor Felix Klein President, Dr. A. Gutzmer (Halle a. S.) Secretary and Treasurer, and appointed Professor A. Wangerin and Dr. Gutzmer a committee to publish the annual volume. The number of members at present is 291.

CAMBRIDGE UNIVERSITY. The lectures on Higher Mathematics to be delivered during the ensuing year include the following subjects:—in *Michaelmas Term*, 1897: Professor STOKES: Connection between certain ascending convergent and descending semi-convergent series.—Professor FORSYTH: Invariants and covariants; Calculus of variations.—Professor

Sir R. S. BALL: Planetary theory.—Dr. HOBSON: Spherical and cylindrical harmonics.—Mr. BAKER: Solid geometry.—Dr. LACHLAN: Projective geometry.—Mr. BERRY: Theory of functions. *Lent Term*, 1898: Professor FORSYTH: Invariants and covariants (ternary forms).—Professor DARWIN; Lunar theory (Hill's method).—Professor BALL: Non-Euclidean geometry.—Mr. PENDLEBURY: Theory of numbers.—Mr. GLAZEBROOK: Sound.—Dr. HOBSON: Higher dynamics.—Mr. BAKER: Abelian functions.—Dr. GLAISHER: Elliptic functions.—Mr. HERMAN: Hydrodynamics.—Mr. BENNETT: Linear and quadratic complexes. *Long Vacation*, 1897 and 1898: Professor DARWIN: Figure of earth and precession; Potential and attractions.

THE UNIVERSITY OF CHICAGO. The mathematical courses to be offered during the summer quarter of 1897 have already been announced in the BULLETIN (p. 293).

During the three quarters (a, w, s) of the academic year 1897–98 the following advanced mathematical courses (four or five hours weekly) will be offered: By Professor MOORE: Seminar devoted to research work, especially in groups, algebra and arithmetic (w, s); Theory of substitutions (a); Theory of numbers (a); General arithmetic (w, s).—By Professor BOLZA: Theory of invariants (a, $\frac{1}{2}$ w); Calculus of variations ($\frac{1}{2}w$); Advanced integral calculus (a, w).—By Associate Professor MASCHKE: Differential geometry (w, s); Functions of a complex variable (w); Linear differential equations (s).—By Dr. HANCOCK: Theory of equations (a, w).—By Dr. LAVES: Analytic mechanics (a, w). The Mathematical Club meets fortnightly.

COLUMBIA UNIVERSITY. During the academic year 1897–98 the following advanced courses will be given by the Department of Mathematics, each course occupying three hours a week: By Professor FISKE: (1) Special topics in differential and integral calculus, first half year; (2) Differential equations with applications, second half year; (3) General theory of functions of a complex variable, first half year; (4) Elliptic functions, second half year.—By Professor COLE: (5) Groups of linear transformations and the theory of the equation of the fifth degree, both half years; (6) Theory of substitutions, both half years.—By Mr. MACLAY: (7) Analytical theory of curves of double curvature and curved surfaces, both half years.—By Dr. CHITTENDEN: (8) Theory of invariants, first half year; (9) Higher plane curves, second half year.

The advanced courses to be given by the Department of

Mechanics are the following: By Professor WOODWARD: (1) Theory of the potential function, with applications to gravitation, hydromechanics, geodynamics, static electricity and terrestrial magnetism, two hours per week throughout the year; (2) Advanced analytical mechanics, embracing the general principles of kinematics, statics and kinetics, with applications of the methods of Lagrange, Hamilton and Jacobi to the interpretation of mechanical phenomena, two hours per week throughout the year.—By Professor PUPIN: (3) General electromagnetic theory, two hours per week throughout the year; (4) Electromagnetic theory of light, three hours per week, second half year.—By Mr. J. C. PFISTER: (5) Theoretical mechanics, two hours per week throughout the year; (6) Elementary thermodynamics and hydromechanics; thermodynamics (based on Maxwell's Theory of Heat), two hours per week, first half year; hydromechanics, two hours per week, second half year.

MR. G. H. HINTON, for the past four years instructor in mathematics at Princeton University, has accepted a call to be assistant professor of mathematics at the University of Minnesota.

THE vacancy among the corresponding members of the Paris Academy of Sciences, caused by the death of Professor SYLVESTER, has been filled by the election of Professor FELIX KLEIN.

THE officers of Section A (mathematical and physical science) for the Toronto meeting of the British Association for the Advancement of Science are: *President*, Professor A. R. FORSYTH; *Vice-Presidents*, R. T. GLAZEBROOK, Professor A. JOHNSON, Professor O. J. LODGE; *Secretaries*, J. C. GLASHAN, Professor W. H. HEATON; *Recorders*, J. L. HOWARD, Professor J. G. MCGREGOR.

DR. FRANZ MEYER, docent in mathematics in the School of Mines at Klausthal, has been called to the University of Königsberg.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLYAI DE BOLYA (W.). Tentamen iuventutem studiosam in elementa matheseos purae elementaris ac sublimioris methodo intuitiva evidentiaque huic propria introducendi, cum appendice triplici. Ed. II. Tom. I: Conspectus arithmeticae generalis. Mandato Academiae scientiarum hungaricae suis adnotationibus adiectis ediderunt J. König et M. Réthy. Budapestini (Berlin, Friedländer), 1897. 4to. 12 and 679 pp. Portrait. Mk. 40.00

BRIGGS (W.) and BRYAN (G. W.). Coördinate geometry: The right line and circle. 2d edition. London, 1897. 8vo. 3s. 6d.

———. Key to coördinate geometry: The right line and circle. London, 1897. 8vo. 5s.

BRYAN (G. W.). See BRIGGS (W.).

HALSTED (G. B.). Sylvester. (*Science*, Vol. V, pp. 597–604.) 4to. 8 pp.

HERRMANN (O.). Über algebraische Kurven, die sich beliebig eng an gegebene Kurvenpolygone anschliessen. Ein Beitrag zur Lehre von der Gestalt algebraischer Kurven. Leipzig, Hinrichs, 1897. 4to. 26 pp. Mk. 1.00

KLEIN (F.). Autographirte Vorlesungshefte III: Ausgewählte Kapitel der Zahlentheorie; zweistündige Vorlesung im Winter 1895–96 und Sommer 1896. (*Mathematische Annalen*, Vol. 48; pp. 562–588.) 4to. 26 pp.

———. On the stability of a sleeping top. Abstract of a lecture before the American Mathematical Society at the Princeton Meeting, October 17, 1896. (*Bulletin of the American Mathematical Society*, 2d series, Vol. III, No. 4, pp. 129–132.) 8vo. 4 pp.

KÖNIG (J.). See BOLYAI DE BOLYA (W.).

KÖSTLIN (W.). Ueber Singularitäten ebener algebraisher Curven. Tübingen, 1895. 8vo. 36 pp. Mk. 1.50

LAMPE (E.). Karl Weierstrass. Gedächtnisrede. Leipzig, Barth, 1897. 8vo. 24 pp. Mk. 0.60

MOSNAT (E.). Problèmes de géométrie analytique à l'usage aux écoles navale, centrale, des ponts et chaussées, des mines et de Saint-Étienne. 2e édition. Paris, Nony, 1897. 8vo. 496 pp. Fr. 6.00

OSGOOD (W. F.). Introduction to infinite series. Cambridge, Harvard University, 1897. 8vo. 71 pp. $0.75

PASCAL (E.). Calcolo delle variazioni e calcolo delle differenze finite. Parte III del calcolo infinitesimale. Milano, Hoepli, 1897. 16mo. 12mo. 12 and 330 pp.

RENNER (L.). Ueber die Gruppe der 24 Collineationen, durch welche ein ebenes Viereck oder Vierkant in sich selbst übergeht. Strassburg, 1896. 8vo. 26 pp. Mk. 1.50

RÉTHY (M.). See BOLYAI DE BOLYA (W.).

SCHLUMBERGER (G.). Ueber n-dimensionale lineare und quadratische Kugelsysteme. Zürich, 1896. 8vo. 6 and 78 pp. Mk. 1.80

SÖDERBLOM (A.) Inledning till Rymden's analytiska Geometri. Göteborg, 1896. 8vo. 58 pp. Mk. 2.50

VAILATI (G.). Sull' importanza delle ricerche relative alla storia delle scienze. Prolusione a un corso sulla storia della meccanica. (Letta il giorno 4 dicembre 1896 nell' Università di Torino.) Torino, Frassati, 1897. 8vo. 22 pp.

II. ELEMENTARY MATHEMATICS.

BRIGGS (W.) and EDMONDSON (T. W.). Mensuration and spherical geometry. Being mensuration of the simpler figures and geometrical properties of the sphere. London, 1897. 8vo. 3s. 6d.

CARLES (J. D.). Aritmética razonada y nociones de algebra. Barcelona, 1897. 4to. 513 pp. Mk. 6.50

EDMONDSON (T. W.). See BRIGGS (W.).

ESTILL (J. G.). Numerical problems in plane geometry with metric and logarithmic tables. New York, London, and Bombay, Longmans, 1897. 8vo. 7 and 144 pp.

EVANS (T. J.) and PULLEN (W. W. F.). Practical plane and solid geometry: a treatise containing solutions to the honors questions at the examinations of the Science and art department (1887-96). London, Chapman, 1897. 8vo. 408 pp. 9 s.

HENSELIN (A.). Rechentafel. Enthaltend das grosse Einmaleins bis 999 mal 999, mit einer Einrichtung, die es ermöglicht, jedes gesuchte Resultat sowohl für die Multiplikation, als auch für die Division blitzschnell zu finden, nebst einer Kreisberechnungstabelle. Berlin, Elsner, 1897. Folio. 3 and 222 pp. Mk. 6.00

JENTZEN. Elemente der Trigonometrie zum praktischen Gebrauch für Unterrichtszwecke an mittleren technischen Lehranstalten. 2te Auflage. Dresden, Kühtmann, 1897. 8vo. 55 pp. Mk. 1.20

KINKELIN (H.) See RIEM (J.).

LONEY (S. L.) See TODHUNTER (I.).

MOCNIK (F. R.). Fünfstellige Logarithmen-Tafeln. Wien und Prag, Tempsky-Freitag, 1887. 8vo. 12 and 71 pp. Mk. 1.20

———. Logarithmisch-trigonometrische Tafeln. 5te Auflage. Wien und Prag, Tempsky-Freitag, 1897. 8vo. 12 and 77 pp. Mk. 1.30

PULLEN (W. W. F.). See EVANS (T. J.)

RIEM (J.). Rechentabellen für Multiplikation und Division, mit einem Vorworte von H. Kinkelin. (Deutsch und französisch.) Basel, Schweizerische Verlags-Druckerei, 1897. 8vo. 12 and 179 pp. Mk. 10.00

ROBBINS (E. R.). Algebra reviews. Boston, Ginn, 1897. 16mo. 4 and 44 pp. $0.27

SEEGER (H.). Die Elemente der Arithmetik. Für den Schulunterricht bearbeitet. 2 Theile. Buch III: Pensum der Ober-Tertia; Buch IV: Pensum der Unter-Sekunda; Buch V: Pensum der Ober-Sekunda. Güstrow, Opitz, 1897. 8vo. 4 and 159 pp. Mk. 2.20

TEEGAN (J. H.). Key to elementary and intermediate algebra. London, Simpkins, 1897. 8vo. 184 pp. 5s.

TODHUNTER (I.) Algebra for beginners. With numerous examples. New edition, revised and enlarged by S. L. Loney. London, Macmillan, 1897. 12mo. 8 and 428 pp. $1.10

WALLENTIN (F.). Maturitätsfragen aus der Mathematik. Zum Gebrauche für die obersten Klassen der Gymnasien und Realschulen zusammengestellt. Auflösungen. 3te Auflage. Wien, Gerold, 1897. 8vo. 6 and 208 pp. Mk. 4.00

III. APPLIED MATHEMATICS.

BOCK (A.). Ueber die Dampfspannung an gekrümmten Flüssigkeits-Oberflächen. Rothenburg, 1896. 8vo. 44 pp. Mk. 1.50

BUCHERER (A. H.). Die Wirkung des Magnetismus auf die elektromotorische Kraft. Strassburg, 1896. 8vo. 17 pp. Mk. 1.20

BUSCH (A.). Ueber oscillatorische Condensator-Entladungen. Freiburg, 1896. 8vo. 46 pp. Mk. 1.50

CARTER (E. T.). Motive power and gearing for electrical machinery. Treatise on the theory and practice of the mechanical equipment of power stations for electrical supply and for electric traction. London, 1896. 8vo. 642 pp. 13s. 6d.

CHOMÉ (F.). Eléments de géométrie descriptive. Bruxelles, 1897. 8vo. 4to. With plates. Fr. 5.00

DEBO (L.). Die Lage der neutralen Schichte bei gebogenen Körpern und die Druckvertheilung im Mauerwerke bei excentrischer Belastung. Hannover, Schmorl, 1897. 8vo. 7 and 87 pp. Mk. 1.80

KAPP (G.). Elektrische Wechselströme. Deutsche Ausgabe von H. Kaufmann. 2te Auflage. Leipzig, Leimer, 1897. 8vo. 92 pp. Mk. 2.75

KAUFMANN (H.). See KAPP (G.).

KRÜGER (S.). Ellipsoidale Gleichgewichtsformen einer rotirenden homogenen Flüssigkeitsmasse. Leiden, 1896. 8vo. 186 pp. Mk. 2.50

KUTTER (W. R.). Bewegung des Wassers in Kanälen und Flüssen. Tabellen und Beiträge zur Erleichterung des Gebrauchs der neuen allgemeinen Geschwindigkeits-Formel von Ganguillet und Kutter. 2te Auflage. 2ter Abdruck. Berlin, Parey, 1897. 8vo. 4 and 134 pp. Mk. 7.00

MACH (E.). Die Mechanik in ihrer Entwickelung, historisch-kritisch dargestellt. 3te vermehrte und verbesserte Auflage. Leipzig, Brockhaus, 1897. 8vo. Mk. 8.00

MÜLLER (O.). Hilfstafeln für praktische Messkunde, nebst logarithmisch-trigonometrischen Tafeln. Zürich, Schulthess, 1896. 8vo. 144 pp. Mk. 2.40

PRESTON (E. D.). A graphic method of reducing stars from mean to apparent places. (U. S. Coast and Geodetic Survey, Report for 1895, Appendix No. 7, pp. 371–380.) Washington, Government Printing Office, 1896. 4to. 10 pp.

TROWBRIDGE (J.). What is electricity? New York, 1896. 8vo. 5 and 315 pp. Mk. 7.50

WEBSTER (A. G.). The theory of electricity and magnetism; being lectures on mathematical physics. New York, Macmillan, 1897. 8vo. 10 and 576 pp. $3.50

WEDELL-WEDELLSBORG (P. S.). Julius Thomsen's Dualismus der chemischen Masse, beleuchtet durch Aufstellung einer neuen Wärmetheorie. Vorläufige Skizze. Kopenhagen, Höst, 1897. 4to. 11 pp. Mk. 1.00

WINTER (W.). Lehrbuch der Physik. 4te Auflage. München, 1897. 8vo. 8 and 521 pp. Mk. 4.80

WRIGHT (T. W.). Elements of Mechanics, including kinematics, kinetics and statics, with applications. New York, Van Nostrand, 1896. 8vo. 8 and 372 pp. Cloth. $2.50